cse581 DM DEFINITIONS 1

PART 1: SETS AND OPERATIONS ON SETS

Subset Notations We use notation $A \subseteq B$ for a SUBSET (might be improper) and $A \subset B$ for a PROPER subset.

Set Inclusion $A \subseteq B$ iff $\forall a(a \in A \Rightarrow a \in B)$ is a true statement.

Set Equality $A = B$ iff $A \subseteq B \cap B \subseteq A$.

Proper Subset $A \subset B$ iff $A \subseteq B \cap A \neq B$.

Power Set $\mathcal{P}(A) = \{B : \ B \subseteq A\}$.

Union $A \cup B = \{x : \ x \in A \cup x \in B\}$. We write:

$x \in (A \cup B)$ iff $x \in A \cup x \in B$.

Intersection $A \cap B = \{x : x \in A \cap x \in B\}$. We write:

$x \in (A \cap B)$ iff $x \in A \cap x \in B$.

Relative Complement $A - B = \{x : x \in A \cap \ x \notin B\}$. We write:

$x \in (A - B)$ iff $x \in A \cap \ x$
$not \in B$.

Complement This is defined only for $A \subseteq U$, where U is called an UNIVERSE.

We define: $-A = U - A$, or write: $x \in -A$ iff $x \notin A$.

Other notation some books use A^c, or $'$ for $-A$.

Set A defined by **a property (predicate) P(x)** is $A = \{x : \ P(x)\}$.

Ordered Pair Given two sets A, B, we denote by **(a, b)** and ordered pair, where $a \in A$ and $b \in B$. a is a first coordinate, b is the second coordinate. We define:

$(a, b) = (c, d)$ iff $a = c$ and $b = d$.

(Cartesian) Product of two sets A and B.

$A \times B = \{(a, b) : \ a \in A \cap b \in B\}$, or we write:

$(a, b) \in (A \times B)$ iff $a \in A \cap b \in B$.

Binary Relation R defined in a set A is any subset R of a cartesian product of $A \times A$, i.e. $R \subseteq A \times A$.

Domain of R Let $R \subseteq A \times A$, we define domain of R:

$D_R = \{a \in A : \ \exists b \in A((a, b) \in R)\}$.

Range of R (Set of values of R). Let $R \subseteq A \times A$, we define range of R (set V_R of values of R):

$V_R = \{b \in A : \ \exists a \in A((a, b) \in R)\}$.

Ordered tuple Given sets $A_1, ...A_n$. An element $(a_1, a_2, ...a_n)$ such that $a_i \in A_i$ for $i = 1, 2, ...n$ is called an ordered TUPLE.

(Cartesian) Product of sets $A_1, ...A_n$.
$A_1 \times A_2 \times ... \times A_n = \{(a_1, a_2, ...a_n) : \ a_i \in A_i, \ i = 1, 2, ...n\}$.

Algebra of sets consists of properties of sets that are TRUE for ALL sets involved. We use tautologies of propositional logic to prove BASIC properties of sets and we use the basic properties to prove more elaborated properties of set

PART 2: FUNCTIONS

Function as Relation $R \subseteq A \times B$ is a FUNCTION from A to B iff
$\forall a \in A \ \exists! \ b \in B \ (a, b) \in R$.

Where $\exists! \ b \in B$ means there is EXACTLY one $b \in B$. Because for all $a \in A$ we have exactly one $b \in B$, we write it as: $a = R(b)$ for $(a, b) \in R$.

A is called A DOMAIN of a function R and we write:
$R : \ A \longrightarrow B$ to denote that $R \subseteq A \times B$ is a FUNCTION from A to B.

Function notation We denote relations that are functions by letters f, g, h,... and write $f : A \longrightarrow B$ to say that $f \subseteq A \times B$ is a function from A to B (**MAPS A into B**).

Domain, codomain of f Let $f : \ A \longrightarrow B$, A is called a DOMAIN of f and B is called a codomain of f.

Graph of f In our approach the GRAPH and the function are the same. **GRAPH**$f = f = \{(a, b) : b = f(a)\}$.

ONTO function $f : \ A \longrightarrow B$ is called an **onto** function and denoted by
$f : \ A \stackrel{onto}{\longrightarrow} B$ iff $\forall b \in B \exists a \in A \ f(a) = b$.

1-1 function $f : \ A \longrightarrow B$ is called a ONE-TO ONE function and denoted by

$f : \ A \stackrel{1-1}{\longrightarrow} B$ iff $\forall x, y \in A (x \neq y \Rightarrow f(x) \neq f(y) \ )$.

f is NOT 1-1 $f : \ A \longrightarrow B$ is **not a ONE-TO ONE function** iff $\exists x, y \in A (x \neq y \cap f(x) = f(y) \ )$.

1-1, onto If f is a **1-1 and onto** function we write it as $f : \ A \stackrel{1-1, onto}{\longrightarrow} B$.

Composition Let $f : \ A \longrightarrow B$ and $g : \ B \longrightarrow C$, we define a new function $h : \ A \longrightarrow C$, called a **COMPOSITION of f and g**, as follows:

for any $x \in A$, $h(x) = g(f(x))$.

Composition notation We denote a composition h of f and g as $h = f \circ g$. I.e. we define: for all $x \in A$, $(f \circ g)(x) = g(f(x))$.

Observe Standard **notation** for a composition of f and g is $f \circ g$.

It means that f is the first function $f : \ A \longrightarrow B$ and g is the second function $g : \ B \longrightarrow C$ and the composition is a function with a **"name"** $f \circ g$ which is defined by a formula:

for all $x \in A$, $(f \circ g)(x) = g(f(x))$.

Inverse function Let $f : \ A \longrightarrow B$ and $g : \ B \longrightarrow A$.

The function g is called an **INVERSE** function to f iff the composition of f and g is an identity on A, i.e. the following condition holds.

$$\forall a \in A, \ (f \circ g)(a) = g(f(a) = a).$$

Inverse function notation If g is an INVERSE function to f we denote by $g = f^{-1}$.

Identity function $f: \ A \longrightarrow A$ is called an IDENTITY on A iff $\forall a \in A f(a) = a$.

Inverse and Identity Let $f: \ A \longrightarrow B$ and $f^{-1}: \ B \longrightarrow A$ is an inverse to f, then the compositions $f \circ f^{-1}$ and $f^{-1} \circ f$ are both identities on A and B, respectively, i.e.

$(f \circ f^{-1})(a) = f^{-1}(f(a)) = a$, for all $a \in A$

and $(f^{-1} \circ f)(b) = f(f^{-1}(b)) = b$ for all $b \in B$.

Inverse Function Theorem For any function $f: \ A \longrightarrow B$, the inverse function to f **exists** iff f is $1-1$ and ONTO, i.e. $f: \ A \overset{1-1,onto}{\longrightarrow} B$.

PART 3: SEQUENCES, GENERALIZED UNION AND INTERSECTION

A sequence of elements of a set A is any function

$f: N \longrightarrow A$ or $f: N - \{0\} \longrightarrow A$.

n-th term of a sequence Let $f: \ N \longrightarrow A$ be a sequence, $a_n = f(n)$ is called a n-th term of a sequence f and we write the sequence f as $a_0, a_1, ...a_n, ...$

Sequence notation Let f be a sequence, we denote it as $\{a_n\}_{n \in N}$, or $\{a_n\}_{n \in N - \{0\}}$.

Finite Sequence of elements of a set A is any function $f: \ \{1, 2, ...n-1\} \longrightarrow A$, for $n \in N$ and n is called a LENGTH of the sequence f. Observe that for $n = 0$, $f = \emptyset$ and we call the sequence of lenght 0 the empty sequence, and denote by e or λ.

Family of sets Any collection of sets is called a Family of sets. We denote it by $\mathcal{F}$.

Sequence of sets is a sequence $f: \ N \longrightarrow \mathcal{F}$, i.e a sequence where all its elements are SETS.

We use CAPITAL letters to denote the sets, so we also use capital letters to denote sequences of sets: $\{A_n\}_{n \in N}$, or $\{A_n\}_{n \in N - \{0\}}$.

Generalized Union of a sequence of sets: $\bigcup_{n \in N} A_n = \{x : \ \exists n \in N \ x \in A_n\}$, i.e. $x \in \bigcup_{n \in N} A_n$ iff $\exists n \in N \ x \in A_n$.

Generalized Intersection of a sequence of sets:

$\bigcap_{n \in N} A_n = \{x : \ \forall n \in N \ x \in A_n\}$, i.e.

$x \in \bigcap_{n \in N} A_n$ iff $\forall n \in N \ x \in A_n$.

Indexed Family of Sets Let $\mathcal{F}$ be a family of sets, and $T \neq \emptyset$.

Any $f: \ T \longrightarrow \mathcal{F}$, $f(t) = A_t$ is called an indexed family of sets, T is called a set if indexes.

We write it: $\{A_t\}_{t \in T}$.

NOTICE that any sequence of sets is an indexed family of sets for $T = N$.

Generalized Union of an indexed family of sets:

$\bigcup_{t \in T} A_t = \{x : \ \exists t \in T \ x \in A_t\}$, i.e. $x \in \bigcup_{t \in T} A_t$ iff $\exists t \in T \ x \in A_t$.

Generalized Intersection of an indexed family of sets: $\bigcap_{t \in T} A_t = \{x : \ \forall t \in T \ x \in A_t\}$, i.e. $x \in \bigcap_{t \in T} A_t$ iff $\forall t \in T \ x \in A_t$.

PART 4: IMAGE AND INVERSE IMAGE

Image of a set $A \subseteq X$ under a function $f: \ X \longrightarrow Y$. NOTATIONS: $f(A)$ or $f^{\rightarrow}(A)$. Definition:
$f(A) = f^{\rightarrow}(A) = \{y \in Y : \ \exists x(x \in A \cap y = f(x))\}$, i.e.

$y \in f(A)$ iff $\exists x(x \in A \cap y = f(x))$.

Inverse Image of a set $B \subseteq Y$ under a function $f: \ X \longrightarrow Y$. NOTATIONS: $f^{-1}(B)$ or $f^{\leftarrow}(B)$. Definition:

$f^{-1}(B) = f^{\leftarrow}(B) = \{x \in X : \ f(x) \in B\}$, i.e.

$x \in f^{-1}(B)$ iff $f(x) \in B$.

PART 5: EQUIVALENCE, PARTITION

Equivalence relation $R \subseteq A \times A$ is an equivalence relation in A iff it is relexive, symmetric and transitive.

Equivalence relation symbols We denote equivalence relation by $\sim$, or $\approx$, or $\equiv$. In my notes we usually use $\approx$ as a symbol for the equivalence relation.

Equivalence class If $\approx \subseteq \ A \times A$ is and equivalence relation then the set

$E = \{b \in A : \ a \approx b\}$ is called an equivalence class.

Equivalence class symbols The equivalence classes are usually denoted by:

$[a] = \{b \in A : \ a \approx b\}$

and the element a is called a **representative of the equivalence class**

$[a] = \{b \in A : \ a \approx b\}$.

Other symbols used are: $|a|$ or $\| a \|$ for the equivalence class $\{b \in A : \ a \approx b\}$ with a representative a.

Partition A family of sets $\mathbf{P} \subseteq \mathcal{P}(A)$ is called a partition of the set A iff the following conditions hold.

1. $\forall \ X \in \mathbf{P} \ (X \neq \emptyset)$
   i.e. all sets in the partion are non-empty.
2. $\forall \ X, Y \in \mathbf{P} \ (X \cap Y = \emptyset)$
   i.e. all sets in the partion are disjoint.
3. $\bigcup \mathbf{P} = A$
   i.e sum of all sets from $\mathbf{P}$ is the set A.

Notation: $A/\approx$ denotes the set of all equivalence classes of $\approx$, i.e.

$A/\approx = \{[a] : a \in A\}$.

Equivalence and Partition Theorem

Let $A \neq \emptyset$, if $\approx$ is an equivalence relation on A, then $A/\approx$ is a partition of A, i.e.

1. $\forall \ [a] \in A/\approx \ \ ([a] \neq \emptyset)$
   i.e. all equivalence classes are non-empty.
2. $\forall \ [a] \neq [b] \in A/\approx \ \ ([a] \cap [b] = \emptyset)$
   i.e. all equivalence classes are disjoint.
3. $\bigcup A/\approx = A$
   i.e sum of all equivalence classes (sets from $A/\approx$) is the set A.

Partition and Equivalence We prove also a following:

For partition $\mathbf{P} \subseteq \mathcal{P}(A)$ of A, there is an equivalence relation on A such that its equivalence classes are exactly the sets of the partition $\mathbf{P}$.

Sets R(a) Observe that we can consider, for ANY relation R on A sets that "look" like equivalence classes i.e. are defined as follows:

$R(a) = \{b \in A; \quad aRb\} = \{b \in A; \quad (a, b) \in R\}$.

Fact 1 If R is an equivalence on A, then the family $\{R(a)\}_{a \in A}$ is a partition of A.

Fact 2 If the family $\{R(a)\}_{a \in A}$ is NOT a partition of A, then R is NOT an equivalence on A.