Chapter 1
Introduction: Paradoxes and Puzzles

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PART 1: Logic for Mathematics
Logical Paradoxes
Logical Paradoxes

Early intuitive approach
Till the end of the 19th century, mathematical theories used to be built in an intuitive, not formal axiomatic way.

Historical development of mathematics has shown that it is not sufficient to base mathematical theories only on an intuitive understanding of their notions, as the following historical example shows.
Example

By a **set**, we mean **intuitively**, any **collection** of objects

For example, the **set** of all **even integers** or the **set** of all **students** in a class

The **objects** that make up a **set** are called its **members** (elements)

**Sets** may themselves be **members** of **sets**
For example, the **set of all subsets** of **integers** has **sets** as its **members**
Example

Most *sets* are not *members* of *themselves*.

The *set* of all *students*, for example, is not a *member* of *itself*.

The *set* of all *students* is *not* a *student*.

However, there may be *sets* that do *belong* to *themselves*.

For example, the *set* of all *sets*. 
Russell Paradox

Russell Paradox (1902)
Consider the set $A$ of all those sets $X$ such that $X$ is not a member of $X$

Clearly, $A$ is a member of $A$ if and only if $A$ is not a member of $A$
So, if $A$ is a member of $A$, the $A$ is also not a member of $A$; and if $A$ is not a member of $A$, then $A$ is a member of $A$

In any case, $A$ is a member of $A$ and $A$ is not a member of $A$

Contradiction
Russell Paradox Solution

Russell proposed and developed a **theory of types** as a solution to the Russell Paradox.

The **idea** is that every **object** must have a definite non-negative **integer** as its **type** assigned to it.

An expression: "**x is a member** of the **set** **y**" is **meaningful** if and only if the **type** of **y** is **one greater** than the **type** of **x**.
Russell Paradox Solution

Russell’s theory of types guarantees that it is meaningless to say that a set belongs to itself.

Hence Russell’s solution is:
The set $A$ as stated in the Russell Paradox does not exist.

The Type Theory was extensively developed by Whitehead and Russell in years 1910 - 1913.
Logical Paradoxes

Logical Paradoxes, also called Logical Antinomies are paradoxes concerning the notion of a set

A development of Axiomatic Set Theory as one of the most important fields of modern Mathematics, or more specifically of Mathematical Logic or Foundations of Mathematics resulted from the search for solutions to various Logical Paradoxes

First paradoxes free Axiomatic Set Theory was developed by Zermello in 1908
Logical Paradoxes

Two of the most known logical paradoxes (antinomies), other then Russell’s Paradox are those of Cantor and Burali-Forti

They were stated at the end of 19th century

Cantor Paradox involves the theory of cardinal numbers

Burali-Forti Paradox is the analogue to Cantor’s but in the theory of ordinal numbers
Cardinality of Sets

We say that sets $X$ and $Y$ have the same cardinality, $\text{card}X = \text{card}Y$, or that they are equinumerous if and only if there is one-to-one correspondence that maps $X$ onto $Y$.

We say that $\text{card}X \leq \text{card}Y$ if and only if the set $X$ is equinumerous with a subset of the set $Y$.

We say that $\text{card}X < \text{card}Y$ if and only if $\text{card}X \leq \text{card}Y$ and $\text{card}X \neq \text{card}Y$. 
Cantor and Schröder- Berstein Theorems

Cantor Theorem
For any set \( X \),
\[ \text{card}X < \text{card}\mathcal{P}(X) \]

Schröder- Berstein Theorem
For any sets \( X \) and \( Y \),
If \( \text{card}X \leq \text{card}Y \) and \( \text{card}Y \leq \text{card}X \), then
\[ \text{card}X = \text{card}Y \]
Cantor Paradox (1899)

Let $C$ be the universal set - that is, the set of all sets

Now, $\mathcal{P}(C)$ is a subset of $C$, so it follows easily that

$\text{card}\mathcal{P}(C) \leq \text{card}C$

On the other hand, by Cantor Theorem,

$\text{card}C < \text{card}\mathcal{P}(C) \leq \text{card}\mathcal{P}(C)$, so also $\text{card}C \leq \text{card}\mathcal{P}(C)$

From Schröder- Berstein theorem we have that

$\text{card}\mathcal{P}(C) = \text{card}C$, what contradicts Cantor Theorem

Solution: Universal set does not exist.
Burali-Forti Paradox

Ordinal numbers are special measures assigned to ordered sets

Burali-Forti Paradox (1897)
Given any ordinal number, we know that there is a still larger ordinal number

But the ordinal number determined by the set of all ordinal numbers is the largest ordinal number

Solution: the set of all ordinal numbers do not exist
Logical Paradoxes

Another solution to Logical Paradoxes is to reject the assumption that for every property $P(x)$, there exists a corresponding set of all objects $x$ that satisfy $P(x)$.

The Russell’s Paradox then proves that there is no set $A$ defined by a property $P(X)$: $X$ is a set of all sets that do not belong to themselves.
Logical Paradoxes

**Cantor Paradox** shows that there is no set $A$ defined by a property $P(X)$: there is an universal set $X$

**Burali-Forti Paradox** shows that there is no set $A$ defined by a property $P(X)$: there is a set $X$ that contains all ordinal numbers
Intuitionism

A more radical interpretation of the paradoxes has been advocated by Brouwer and his intuitionist school. Intuitionists refuse to accept the universality of certain basic logical laws, such as the law of excluded middle:

- A or not A

For Intuitionists, the excluded middle law is true for finite sets, but it is invalid to extend it to all other sets.

The Intuitionists’ concept of infinite set differs from that of classical mathematicians.
Intuitionists’ Mathematics

The basic difference between classical and intuitionists’ mathematics lies also in the interpretation of the word exists.

In classical mathematics proving existence of an object $x$ such that $P(x)$ holds does not mean that one is able to indicate a method of construction of it.

In the intuitionists’ universe we are justified in asserting the existence of an object having a certain property only if we prove existence of an effective method for constructing, or finding such an object.
Intuitionists’ Mathematics

In intuitionistic mathematics the logical paradoxes are not derivable, or even meaningful.

The Intuitionism, because of its constructive flavor, has found a lot of applications in computer science, for example in the theory of programs correctness.

Intuitionistic Logic (to be studied in the book) reflects intuitionists ideas in a form a formalized deductive system.
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PART 2: Logic for Mathematics
Semantic Paradoxes
Semantic Paradoxes

The development of axiomatic theories solved some, but not all problems brought up by the Logical Paradoxes.

Even the consistent sets of axioms, as the following examples show, do not prevent the occurrence of another kind of paradoxes, called Semantic Paradoxes.

The Semantic Paradoxes deal with the notion of truth.
Semantic Paradoxes

Berry Paradox, 1906:
Let $A$ denote the set of all positive integers which can be defined in the English language by means of a sentence containing at most 1000 letters.

The set $A$ is finite since the set of all sentences containing at most 1000 letters is finite. Hence, there exist positive integers which do not belong to $A$.

Consider a sentence: $n$ is the least positive integer which cannot be defined by means of a sentence of the English language containing at most 1000 letters.

This sentence contains less than 1000 letters and defines a positive integer $n$.

Therefore $n \in A$ - but $n \notin A$ by the definition of $n$.

CONTRADICTION!
Berry Paradox Analysis

The paradox resulted entirely from the fact that we did not say precisely what notions and sentences belong to the arithmetic and what notions and sentences concern the arithmetic.

Of course we didn’t talk about and examine arithmetic as a fix and closed deductive system.

We also incorrectly mixed the natural language with mathematical language of arithmetic.
Berry Paradox Solution

We have to always clearly distinguish between the language of the theory (arithmetic) and the language in which we talk about the theory, which is called a metalanguage.

In general we must clearly distinguish a formal theory from the meta-theory.

In well and correctly defined theory such paradoxes can not appear.
The Liar Paradox

Liar Paradox

A man says: I am lying
If he is lying, then what he says is true, and so he is not lying
If he is not lying, then what he says is not true, and so he is lying

Contradiction
Liar Paradoxes

These **paradoxes arise** because the **concepts** of the type

“**I am true**”, “**this sentence is true**”, “**I am lying**”

should not occur in the **language** of the **theory**

They belong to a **metalanguage** of the **theory**

It it means they belong to a **language** that talks **about** the **theory**
Cretan Paradox

The Liar Paradox is a corrected version of a following paradox stated in antiquity by a philosopher Epimenides

Cretan Paradox
The Cretan philosopher Epimenides said:
All Cretans are liars
If what he said is true, then, since he is a Cretan, it must be false and what he said is false
Thus, there is a Cretan who is not a liar

Contradiction
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PART 3: Logics for Computer Science:
Classical, Intuitionistic, Modal, Temporal, Many Valued
Classical and Intuitionistic Logics

The use of Classical Logic in computer science is known, indisputable, and well established.

The existence of PROLOG and Logic Programming as a separate field of computer science is the best example of it.

Intuitionistic Logic in the form of Martin-Löf’s theory of types (1982), provides a complete theory of the process of program specification, construction, and verification.

A similar theme has been developed by Constable (1971) and Beeson (1983)
Modal Logics

Modal Logic was created by C.I. Lewis in 1918

In an attempt to avoid, what some felt, the paradoxes of classical implication (a false sentence implies any sentence) he proposed a new interpretation of the logical implication

The idea was to distinguish two sorts of truth: necessary truth and mere possible truth

As a consequence a new, modal logic was created
Modal Logics for Computer Science

Modal Logics in Computer Science are used as a tool for analyzing such notions as knowledge, belief, tense.

Modal logics have been also employed in a form of Dynamic logic (Harel, 1979) to facilitate the statement and proof of properties of programs.
Temporal Logics were created for the specification and verification of concurrent programs by Harel (1979) and Parikh (1983).

For a specification of hardware circuits by Halpern, Manna, Maszkowski (1983).

Temporal Logics were also used to specify and clarify the concept of causation and its role in commonsense reasoning by Shoham (1988).
Other Non-classical Logics

The development of new logics and the applications of logics to different areas of Computer Science and in particular to Artificial Intelligence is a subject of a book in itself but is beyond the scope of this book.

The book examines in detail the classical logic and some aspects of the intuitionistic logic and its relationship with the classical logic.

It introduces some of the most standard many valued logics, and examines modal S4, S5 logics. It also shows the relationship between the modal S4 and the intuitionistic logics.
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PART 4: Computer Science Puzzles
Reasoning in Artificial Intelligence
Reasoning in Distributive Systems


Two divisions of an army are camped on two hilltops overlooking a common valley.

In the valley awaits the enemy.

If both divisions attack the enemy simultaneously, they will win the battle.

If only one division attacks, it will be defeated.
Coordinated Attack

The divisions do not initially have plans for launching an attack on the enemy.

The commanding general of the first division wishes to coordinate a simultaneous attack. Neither general will decide to attack unless he is sure that the other will attack with him.

The generals can only communicate by means of a messenger.
Coordinated Attack

It takes a messenger one hour to get from one encampment to the other. However, it is possible that the messenger will get lost in the dark or, worst yet, be captured by the enemy.

Fortunately on this particular night, everything goes smoothly.

Question
How long will it take them to coordinate an attack?
Coordinated Attack

Suppose the messenger sent by General A makes it to General B with a message saying Attack at dawn.

Will General B attack?

No, since General A does not know General B got the message, and thus may not attack.
Coordinated Attack

General B sends the messenger back with an acknowledgment.

Suppose the messenger makes it.

Will General A attack?

**No**, because now A is worried that General B does not know A got the message, that General B thinks A may think that B did not get the original message, and thus General A does not attack.
Coordinated Attack

General A sends the messenger back with an acknowledgment. This is not enough.

No amount of acknowledgments sent back and forth will ever guarantee agreement.
Even in a case that the messenger succeeds in delivering the message every time.

All that is required in this (informal) reasoning is the possibility that the messenger does not succeed.
Coordinated Attack Solution

To solve this problem Halpern and Moses (1985) created a propositional modal logic with m agents.

They proved this logic to be essentially a multi-agent version of the standard modal logic S5.

They also proved that formally defined common knowledge is not attainable in systems where communication is not guaranteed.
Communication in Distributed Systems

The **common knowledge** is also **not attainable** in systems where communication is **guaranteed**, as long as there is some **uncertainty** in massage delivery time.

In distributed systems where communication is **not guaranteed** **common knowledge** is **not attainable**.

But we often **do reach** agreement!
Communication in Distributed Systems

They proved that formally defined common knowledge is attainable in such models of reality where we assume, for example, events can be guaranteed to happen simultaneously.

Moreover, there are some variants of the definition of common knowledge that are attainable under more reasonable assumptions.

So, we can formally prove that in fact we often do reach agreement!
Reasoning in Artificial Intelligence

**Assumption 1:**
Flexibility of reasoning is one of the key property of intelligence

**Assumption 2:**
Commonsense inference is *defeasible* in its nature; we are all capable of drawing conclusions, acting on them, and then *retracting* them if necessary in the face of new evidence
If computer programs are to act \textit{intelligently}, they will need to be similarly \textit{flexible}.

\textbf{Goal:} development of \textit{formal systems} (logics) that describe commonsense flexibility.
Flexible Reasoning

Example: Reiter, 1987
Consider a statement Birds fly Tweety, we are told, is a bird. From this, and the fact that birds fly, we conclude that Tweety can fly.

This conclusion is defeasible: Tweety may be an ostrich, a penguin, a bird with a broken wing, or a bird whose feet have been set in concrete.

This is a non-monotonic reasoning: on learning a new fact (that Tweety has a broken wing), we are forced to retract our conclusion (that he could fly).
Non-Monotonic and Default Reasoning

Definition
A non-monotonic reasoning is a reasoning in which the introduction of a new information can invalidate old facts.

Definition
A default reasoning (logic) is a reasoning that let us draw plausible inferences from less-than-conclusive evidence in the absence of information to the contrary.

Observe that non-monotonic reasoning is an example of default reasoning.
Believe Reasoning

Example  Moore, 1983
Consider my reason for believing that I do not have an older brother

It is surely not that one of my parents once casually remarked, you know, you don’t have any older brothers, nor have I pieced it together by carefully sifting other evidence

I simply believe that if I did have an older brother I would know about it; therefore since I don’t know of any older brothers of mine, I must not have any
Auto-epistemic Reasoning

The brother **example** reasoning is not **default** reasoning nor **non-monotonic** reasoning. It is a reasoning about one’s own **knowledge** or **belief**.

**Definition**

Any reasoning about one’s own **knowledge** or **belief** is called an **auto-epistemic** reasoning.

**Auto-epistemic** reasoning **models** the reasoning of an ideally rational agent **reflecting upon** his **beliefs** or **knowledge**.

**Logics** which describe it are called **auto-epistemic logics**.
Computer Science Puzzles
Missionaries and Cannibals

Example  McCarthy, 1985

Here is the old Cannibals Problem

Three missionaries and three cannibals come to the river.
A rowboat that seats two is available.
If the cannibals ever outnumber the missionaries on either bank of the river, the missionaries will be eaten.

How shall they cross the river?
Traditional Solution

Traditionally the puzzler is expected to devise a strategy of rowing the boat back and forth that gets them all across and avoids the disaster.

A state is a triple comprising the number of missionaries, cannibals and boats on the starting bank of the river.

The initial state is 331, the desired state is 000.
A solution is given by the sequence:

331, 220, 321, 300, 311, 110, 222, 020, 031, 010, 021, 000
Imagine now giving someone the problem, and after he puzzles for a while, he suggests going upstream half a mile and crossing on a bridge.

What a bridge? you say.
No bridge is mentioned in the statement of the problem.
He replies: Well, they don’t say the isn’t a bridge.

So you modify the problem to exclude the bridges and pose it again.
He proposes a helicopter, and after you exclude that, he proposes a winged horse....
Missionaries and Cannibals Revisited

So you tell him the solution
He attacks your solution on the grounds that the boat might have a leak
After you rectify that omission from the statement of the problem, he suggests that a sea monster may swim up the river and may swallow the boat

Finally, you must look for a mode of reasoning that will settle his hash once and for all
McCarthy Solution

McCarthy proposes circumscription as a technique for solving his puzzle.

He argues that it is a part of common knowledge that a boat can be used to cross the river unless there is something wrong with it or something else prevents using it.

If our facts do not require that there be something that prevents crossing the river, the circumscription will generate the conjecture that there isn’t.
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PART 5: A Short Chapter Overview
Definitions and Facts

Definition
Logical Paradoxes, also called Logical Antinomies are paradoxes concerning the notion of a set.

Definition
Semantic Paradoxes are paradoxes that deal with the notion of truth.

Definition
A non-monotonic inference is a reasoning in which introduction of a new information can invalidate old facts.
Definitions and Facts

Fact
Non-monotonic reasoning is an example of the default reasoning.

Definition
An auto-epistemic reasoning is any reasoning about one’s own knowledge or belief.

Auto-epistemic reasoning models the reasoning of an ideally rational agent reflecting upon his beliefs or knowledge.
Definitions and Facts

Facts
The main difference between classical and intuitionists’ mathematics lies in the interpretation of the word exists.

In classical mathematics proving existence of an object $x$ such that a property $P(x)$ holds does not always mean that one is able to indicate a method of its construction.

In the intuitionists’ universe we are justified in asserting the existence of an object having a certain property only if we know an effective method for constructing, or finding such an object.