

CSE581 MIDTERM 1 SOLUTIONS Fall 2025
(100pts)

MIDTERM 1 has three parts: **PART 1: PROBLEMS** (80pts), **PART 2: Yes/No Questions** (10pts), and **PART 3: Yes/No DM Definitions and Facts** (10pts).

PART 1: PROBLEMS (80pts)

PROBLEM 1 (10pts) Write the natural language statement:

From the fact that there is a bird that does not fly and $4 + 4 = 4$, we deduce the following:

is not possible that all birds fly or it is not necessary that $4 + 4 = 4$.

in a three ways.

WAY 1 (2pts) As a formula $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \square, \diamond, \cap, \cup, \Rightarrow\}}$. Use Propositional Variables a, b, c for consecutive statements, where

a denotes statement: *there is a bird that does not fly*

b denotes statement: $4 + 4 = 4$

c denotes statement: *textit all birds fly*

Formula $A_1 \in \mathcal{F}_1$ is: $((a \cap b) \Rightarrow (\neg \diamond c \cup \neg \square b))$

WAY 2 (3pts) As a formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$. Use Propositional Variables a, b, c, d for consecutive statements, where

a denotes statement: *there is a bird that does not fly*

b denotes statement: $4 + 4 = 4$

c denotes statement: *possible that all birds fly*

d denotes statement: *necessary that $4 + 4 = 4$*

Formula $A_2 \in \mathcal{F}_2$ is: $(a \cap b) \Rightarrow (\neg c \cup \neg d)$

WAY 3 (5pts) As a formula $A_3 \in \mathcal{F}_3$ of a PREDICATE LANGUAGE language $\mathcal{L}_{CON}(\mathbf{P}, \mathbf{F}, \mathbf{V})$ with the set $CON = \{\neg, \square, \diamond, \cap, \cup, \Rightarrow\}$ of propositional connectives.

Use the following Predicates, Functions and Constants: $B(x)$ for x is a bird, $F(x)$ for x can fly, $E(x, y)$ for $x = y$, $f(x, y)$ for $+$, and c for 4.

1. (1pts) (Atomic formulas are: $B(x), F(x), E(f(c, c), c)$)

2. (2pts) Restricted domain formula is:

$$((\exists_{B(x)} \neg F(x) \cap E(f(c, c), c)) \Rightarrow (\neg \diamond \forall_{B(x)} F(x) \cup \neg \square E(f(c, c), c)))$$

3. (2pts) Formula $A_3 \in \mathcal{F}_3$ is:

$$((\exists x (B(x) \cap \neg F(x)) \cap E(f(c, c), c)) \Rightarrow (\neg \diamond \forall x (B(x) \Rightarrow F(x)) \cup \neg \square E(f(c, c), c)))$$

PROBLEM 2 (5pts)

1. Given a classical semantics for $\mathcal{L}_{\{\neg, \cup, \cap, \Rightarrow\}}$. Use "proof by contradiction" method to prove whether

$$\models ((a \cap \neg b) \Rightarrow ((a \cap \neg b) \Rightarrow (\neg d \cup e)))$$

You get 0pts if you use any other method

Solution This is a question about the knowledge of the "proof by contradiction" method.

Assume $\not\models ((a \cap \neg b) \Rightarrow ((a \cap \neg b) \Rightarrow (\neg d \cup e)))$, i.e. there is $v : VAR \rightarrow \{T, F\}$ such that

$$((a \cap \neg b) \Rightarrow ((a \cap \neg b) \Rightarrow (\neg d \cup e))) = F.$$

This is possible if and only if $(a \cap \neg b) = T$ and $((a \cap \neg b) \Rightarrow (\neg d \cup e)) = F$.

$(a \cap \neg b) = T$ if and only if $a = T, \neg b = T$, i.e. $a = T, b = F$.

We evaluate now whether $((a \cap \neg b) \Rightarrow (\neg d \cup e)) = T \Rightarrow (\neg d \cup e) = F$.

This is possible for $d = T$ and $e = F$.

There is **no contradiction** and we proved that any v such that $a = T, b = F, c = F$ is a counter-model for our formula, i.e. we proved that $\not\models ((a \cap \neg b) \Rightarrow ((a \cap \neg b) \Rightarrow (\neg d \cup e)))$

PROBLEM 3 (15pts)

1. (5pts) Prove by stating a mathematical statement **S** about natural numbers N written with logical symbols that provides its counter model (proper implication in one direction) that $(\exists x A(x) \cap \exists x B(x)) \not\equiv \exists x (A(x) \cap B(x))$.

Solution

A following mathematical statement stated in English

There is a natural number $n > 1$ and there is a natural number $n < 1$, but there is no natural number $n, n > 0$ and $n < 0$. contradicts obviously that for any formulas $A(x), B(x) \in \mathcal{F}$,

$$\models ((\exists x (A(x) \cap B(x)) \Rightarrow (\exists x A(x) \cap \exists x B(x))).$$

To prove it **formally** we first write it as a mathematical statement **S** with logical symbols is follows.

$$\mathbf{S} \quad ((\exists_{n \in N} n > 1 \cap \exists_{n \in N} n < 1) \Rightarrow \exists_{n \in N} (n > 1 \cap n < 1))$$

2. (5pts) Write your statement **S** as a well formed formula A of the classical predicate language. Use $N(x)$ to denote $x \in N$, define your symbols for the predicates, functions, and constants you need.

Solution

We use $N(x)$ to denote $x \in N$, $G(x, y)$ for $x > y$, and $L(x, y)$ for $x < y$, and c for 1.

A non - restricted domain logical formula A is:

$$A : ((\exists x G(x, c) \cap \exists x L(x, c)) \Rightarrow \exists x (G(x, c) \cap L(x, c)))$$

Observe that the restriction of the domain in our counter-example DEFINES the UNIVERSE U and Interpretation I of the **model structure** $\mathbf{M} = (U, I)$ such that $A_I = \mathbf{S}$.

3. (5pts) Define a **model structure** $\mathbf{M} = (N, I)$, such that $\mathbf{M} \models A$, for your formula A , i.e. such that $A_I = \mathbf{S}$.

Solution

We define $\mathbf{M} = (U, I)$ for

$$U = N, \quad G_I : >, \quad L_I : <, \quad c_I : 1.$$

PROBLEM 4 (20pts)

Let $\mathcal{L} = \mathcal{L}_{\{\neg, \sim, \Rightarrow, \rightarrow\}}$ be a language with one argument connectives \neg, \sim called **strong negation** and **weak negation**, and with two arguments connectives \Rightarrow, \rightarrow called **strong implication** and **weak implication**.

We define a **3 valued** extensional semantics **M** for the language $\mathcal{L}_{\{\neg, \sim, \Rightarrow, \rightarrow\}}$ by **defining** the connectives $\neg, \sim, \Rightarrow, \rightarrow$ as functions on the set $\{F, \perp, T\}$ of 3 logical values as follows. The functions \neg, \Rightarrow **restricted** to the set $\{F, T\}$ are the same as in the **classical case**. We extend \neg, \Rightarrow to the full set $\{F, \perp, T\}$ of logical values as follows:

for **strong negation** we put $\neg \perp = F$, and for **strong implication** we put $x \Rightarrow \perp = F$ for $x = T, F$ and

$$\perp \Rightarrow y = \begin{cases} \perp & \text{if } y = \perp \\ T & \text{otherwise} \end{cases}$$

We define the **weak negation** $\sim: \{T, \perp, F\} \rightarrow \{T, \perp, F\}$ as

$$\sim x = \begin{cases} T & \text{if } x = \perp \\ x & \text{for } x \in \{T, F\} \end{cases}$$

The **weak implication** $\rightarrow: \{T, \perp, F\} \times \{T, \perp, F\} \rightarrow \{T, \perp, F\}$ is defined for all $x, y \in \{T, \perp, F\}$ as

$$x \rightarrow y = \sim(x \Rightarrow y)$$

1. (10pts) Fill in the connectives tables. Remember that connectives \neg, \Rightarrow on set $\{F, T\}$ are the same as **classical** \neg, \Rightarrow .

Solution

\neg	F	\perp	T
	T	F	F

\sim	F	\perp	T
	F	T	T

\Rightarrow	F	\perp	T
F	T	F	T
\perp	T	\perp	T
T	F	F	T

\rightarrow	F	\perp	T
F	T	F	T
\perp	T	T	T
T	F	F	T

2. (2pts) Prove that any truth assignment v such that $v(a) = v(b) = \perp$ is a **M model** for the formula $(\neg a \rightarrow (\sim a \Rightarrow \neg b))$. You **can use** shorthand notation.

Solution

We evaluate our formula for. $a = b = \perp$ i.e. we evaluate

$$v^*((\neg a \rightarrow (\sim a \Rightarrow \neg b))) = \neg \perp \rightarrow (\sim \perp \Rightarrow \neg \perp) = F \rightarrow (T \Rightarrow F) = F \rightarrow F = T.$$

We write it symbolically $v \models_{\mathbf{M}} (\neg a \rightarrow (\sim a \Rightarrow \neg b))$.

This is not the only model. Find, as an exercise other models.

3. (4pts) We know that in Lukasiewicz semantics $\not\models_{\mathbf{L}} (a \Rightarrow a)$ and $\not\models_{\mathbf{L}} (a \Rightarrow \neg\neg a)$

Use **shorthand notation** to prove that $\not\models_{\mathbf{M}} (a \Rightarrow a)$ and $\models_{\mathbf{M}} (a \Rightarrow \neg\neg a)$.

Solution

To prove $\not\models_{\mathbf{M}} (a \Rightarrow a)$ we have to find a counter MODEL v for it.

Consider any $v: VAR \rightarrow \{F, \perp, T\}$ such that $v(a) = \perp$.

We evaluate $\perp \Rightarrow \perp = F$ and so $(a \Rightarrow a)$ is not a **M** tautology.

To prove that $\models_{\mathbf{M}} (a \Rightarrow \neg\neg a)$ we first observe that it is a classical tautology and the **M** connectives \neg, \Rightarrow on set $\{F, T\}$

are the same as **classical** \neg, \Rightarrow , so to prove $\models_{\mathbf{M}} (a \Rightarrow \neg\neg a)$ we have to consider only the case $a = \perp$ and get $\perp \Rightarrow \neg\neg \perp = \perp \Rightarrow \neg F = \perp \Rightarrow T = T$.

This ends the proof.

4. (4pts) Let **T** be a set of classical tautologies, **L** be a set of Lukasiewicz tautologies, and **MT** be a set of all **M** tautologies. **Prove** that $\mathbf{T} \cap \mathbf{MT} \neq \emptyset$, $\mathbf{L} \neq \mathbf{MT}$ and that the semantics **M** is **well defined**.

Solution

(1pt) We just proved that the formula $(a \Rightarrow \neg\neg a) \in \mathbf{T} \cap \mathbf{MT}$, hence $\mathbf{T} \cap \mathbf{MT} \neq \emptyset$.

As we have proved that $\not\models_{\mathbf{M}} (a \Rightarrow a)$, and we know that $(a \Rightarrow a) \in \mathbf{LT}$ we proved that $\mathbf{LT} \neq \mathbf{MT}$.

(3pts) By definition, semantics **M** is **well defined** if and only if $\mathbf{MT} \neq \emptyset$.

This is true as we have already proved that $(a \Rightarrow \neg\neg a) \in \mathbf{MT}$.

PROBLEM 5 (15pts)

Let \approx be an equivalence on the set N of natural numbers as follows.

For any $n, m \in N$,

$$n \approx m \text{ if and only if } (n)_3 = (m)_3$$

where $(n)_3$ denotes a remainder of division n by 3.

1. (10pts) Find all equivalence classes of \approx , i.e. define a general form of the equivalence classes.

Solution

We know that that remainders of division n by 3 are 0, 1, 2 and we know that that any element of an equivalence class can be chosen as its representant, so we choose the reminders 0, 1, 2 as the representants and the definition of the equivalence class.

$$[0] = \{m \in N : m = 3k \text{ for } k \in N\}$$

$$[1] = \{m \in N : m = 3k + 1 \text{ for } k \in N\}$$

$$[2] = \{m \in N : m = 3k + 2 \text{ for } k \in N\}$$

2. (2pts) Show that $[7] = [1]$, $[999] = [0]$ and $[8] = [2]$.

Solution

We evaluate using the definition of the equivalence class

$$[7] = \{m \in N : 7 \approx m\} = \{m \in N : (7)_3 = (m)_3\} = \{m \in N : 1 = (m)_3\}$$

$$[7] = \{m \in N : m = 3k + 1 \text{ for } k \in N\} = \{1, 4, 7, 10, \dots, \dots\} = [1] = [4] = [7] = [10] = \dots \dots$$

$$[999] = \{m \in N : 999 \approx m\} = \{m \in N : (999)_3 = (m)_3\} = \{m \in N : 0 = (m)_3\}$$

$$[999] = \{m \in N : m = 3k \text{ for } k \in N\} = \{0, 3, 9, 12, \dots, \dots\} = [0] = [3] = [9] = [12] = \dots \dots$$

$$[8] = \{m \in N : 8 \approx m\} = \{m \in N : (8)_3 = (m)_3\} = \{m \in N : 2 = (m)_3\}$$

$$[8] = \{m \in N : m = 3k + 2 \text{ for } k \in N\} = \{2, 4, 7, 10, \dots, \dots\} = [1] = [4] = [7] = [10] = \dots \dots$$

3. (3pts) List all elements of the PARTITION $\mathbf{P} = N / \approx$

Solution

$$\mathbf{P} = N / \approx = \{[0], [1], [2]\} = \{E_0, E_1, E_2\}$$

where

$$E_0 = \{m \in N : m = 3k \text{ for } k \in N\}$$

$$E_1 = \{m \in N : m = 3k + 1 \text{ for } k \in N\}$$

$$E_2 = \{m \in N : m = 3k + 2 \text{ for } k \in N\}$$

PROBLEM 6 (15pts)

Let \leq on N be a binary relation defined on N defined below, where \leq is a natural order on N .

$$(n, m) \in \leq \text{ if and only if } n \leq m \text{ for } n, m \in \text{EVEN} \text{ and } n = m \text{ for } n, m \in \text{ODD}$$

1. (5pts) Prove that (N, \leq) is a POSET.

Solution

Observe that $P1 = (\text{EVEN}, \leq)$ and $P2 = (\text{ODD}, =)$ are basic Posets. The $N = \text{EVEN} \cup \text{ODD}$, so the only properties to verify are antisymmetry and transitivity of \leq for all $n, m \in N$, such that $n \in \text{EVEN}$ and $m \in \text{ODD}$. Both of them hold VACUOUSLY- it means by the fact that they represent and implication $F \Rightarrow F = T$.

2. (10pts) Prove that the POSET (N, \leq) has \aleph_0 MAXIMAL and \aleph_0 MINIMAL elements by listing all of them and justifying. You can draw a DIAGRAM.

Solution

The Poset $P1 = (\text{EVEN}, \leq)$ has a smallest element $n = 0$, the Poset $P2 = (\text{ODD}, =)$ has \aleph_0 MAX and \aleph_0 MIN element - because each $n \in \text{ODD}$ is a MAX and MIN element.

The smallest element $n = 0$ of $P1$ becomes a MIN element in $P = (N, \leq)$. So P has \aleph_0 MINIMAL elements : 0 and each $n \in \text{ODD}$, and \aleph_0 MAXIMAL elements: $n \in \text{ODD}$.

PART 2 : Yes/No DM Definitions and Facts (10 pts)

Here are 10 definitions and Facts (1 points each) with provided **Yes/No** answers about their **correctness**.

Some of provided answers are **wrong**. Some are **correct**

Read carefully. **Circle** only provided answers that are **correct**.

1. **"onto" Function** $f : A \xrightarrow{\text{onto}} B$ if and only if $\forall b \in B \exists a \in A f(a) = b$

y

2. **inverse Image** $x \in f^{-1}[B]$ if and only if $f(x) \in B$, for any $f : X \rightarrow Y$ and any $B \subset Y$.

n

3. **Generalized Intersection** Given a sequence $\{A_n\}_{n \in \mathbb{N}}$ of sets, $\bigcap_{n \in \mathbb{N}} A_n = \{x : \forall n \in \mathbb{N} x \in A_n\}$

n

4. **Equivalence Class** If $\approx \subseteq A \times A$ is an equivalence relation then the set

$E = \{b \in A : a \approx b\}$ is called an equivalence class.

y

5. **Partition** A family of sets $\mathbf{P} \subseteq \mathcal{P}(A)$ is called a partition of the set A if and only if

1. $\forall X \in \mathbf{P} (X \neq \emptyset)$, 2. $\forall X, Y \in \mathbf{P} (X \cap Y = \emptyset)$, 3. $\bigcup \mathbf{P} = A$

y

6. **Infinite Posets** It is possible to order an infinite set A in such a way that the poset (A, \leq) has a unique maximal element and no largest element.

y

7. **Lattice** A poset (A, \leq) is a lattice iff For all $a, b \in A$ both $\text{lub}\{a, b\}$ and $\text{glb}\{a, b\}$ exist.

n

8. **Cardinality Aleph zero** We say that a set A has a cardinality \aleph_0 ($|A| = \aleph_0$) if and only if $|A| = |\mathbb{N}|$

n

9. **Finite Set** A set A is finite if and only if $\exists n \in \mathbb{N} \exists f (f : \{0, 1, 2, \dots, n-1\} \xrightarrow{1-1, \text{onto}} A)$, i.e. we say: a set A is finite if and only if $\exists n \in \mathbb{N} (|A| = n)$.

y

10. **Power ($\mathcal{M}^{\mathcal{N}}$)** $\mathcal{M}^{\mathcal{N}} = \text{card}\{f : f : A \rightarrow B\}$, where A, B are such that $|A| = \mathcal{M}, |B| = \mathcal{N}$

y

PART 3 : Yes/No DM Questions (10 pts)

There are 10 Questions; 1 points each. **Circle** your answer. Write proper justification.
No justification, no credit

1. $(N \times Q) \cap (Q \times Q) = N \times Q$

Justify: $N \subseteq Q$

y

2. Let $A = \{\emptyset, \{a\}, \{\emptyset\}, 1, 2\}$, $B = \{\emptyset, \{\emptyset\}, \emptyset\}$. There is a function $f : A \xrightarrow{1-1} B$.

Justify: $|A| = 5, |B| = 2$.

n

3. $\{2, \{\emptyset\}, \emptyset\} \cap 2^{\emptyset} = \emptyset$

Justify: $2^{\emptyset} = \{\emptyset\}$ and $\{2, \{\emptyset\}, \emptyset\} \cap \{\emptyset\} \neq \emptyset$ as $\emptyset \in \{2, \{\emptyset\}, \emptyset\}$

n

4. Let $f : N \times N \rightarrow Z$ be given by a formula $f(n, m) = n - m^2$. f is not 1 - 1 function.

Justify: Take $(1, 0) \neq (2, 1)$. We get $f(1, 0) = 1 = f(2, 1)$. For example, we get $f(m^2, m) = m^2 - m^2 = 0$ for all $m \in N$.

y

5. $\{(\emptyset, \emptyset), (\{2\}, \{2\}), (3, 3)\}$ represents a transitive relation in a set $A = \{3, \{2\}, \emptyset\}$.

Justify: Vacuously true.

y

6. Let $f : N \rightarrow \mathcal{P}(N)$ be given by formula: $f(n) = \{m \in N : m < n^2\}$. We have that $\emptyset \in f(\{0, 1, 2, 3\})$.

Justify: $f(0) = \emptyset$, so $\emptyset \in f(\{0, 1, 2, 3\})$.

y

7. For any $\approx \subset A \times A$, a set $[a] = \{b \in A : a \approx b\}$ is an equivalence class with a representative a .

Justify: true only when \approx is an equivalence relation.

n

8. Let \approx be an equivalence on A . Each equivalence class of \approx is non-empty.

Justify: Equivalence relation is reflexive, i.e. for each $a \in A$, $a \approx a$ and so $a \in \{b \in A : b \approx a\}$.

y

9. The set 2^N is infinitely countable.

Justify: $|N| < |2^N|$ by Cantor Theorem, and N is infinitely countable.

n

10. $\aleph_0^{\aleph_0} = C$ means that there are C possible sequences that can be form out of an infinitely countable set

Justify: By definition, $\aleph_0^{\aleph_0} = \text{card}\{f : f : A \rightarrow B\}$ for $|A| = |B| = \aleph_0$.

y