CSE581
Computer Science Fundamentals: Theory

Professor Anita Wasilewska
P1 LOGIC: LECTURE 7a
Chapter 7
Introduction to Intuitionistic and Modal Logics

PART 5: Introduction to Modal Logics
Algebraic Semantics for modal S4 and S5
Introduction to Modal Logics

The **non-classical** logics can be divided in **two groups**: those that **rival** classical logic and those which **extend it**

The Lukasiewicz, Kleene, and intuitionistic logics are in the **first** group.
The **modal logics** are in the **second** group.

The **rival** logics do **not** differ from classical logic in terms of the **language** employed.

The **rival** logics **differ** in that certain **theorems** or **tautologies** of classical logic are rendered **false**, or **not provable** in them.
Introduction to Modal Logics

The most notorious example of the rival difference of logics based on the same language is the law of excluded middle

\[(A \cup \neg A)\]

This is provable in, and is a tautology of classical logic

But is not provable in, and is not tautology of the intuitionistic logic

It also is not a tautology under any of the extensional logics semantics we have discussed
Introduction to Modal Logics

Logics which extend classical logic sanction all the theorems of classical logic but, generally, supplement it in two ways.

Firstly, the languages of these non-classical logics are extensions of those of classical logic.

Secondly, the theorems of these non-classical logics supplement those of classical logic.
Modal logics are enriched by the addition of two new connectives that represent the meaning of expressions "it is necessary that" and "it is possible that"

We use the notation:

$I$ for "it is necessary that" and
$C$ for "it is possible that"

Other notations commonly used are:
$\nabla$, $N$, $L$ for "it is necessary that" and
$\Diamond$, $P$, $M$ for "it is possible that"
Introduction to Modal Logics

The symbols \( N, L, P, M \) or alike, are often used in computer science.

The symbols \( \nabla \) and \( \Diamond \) were *first* to be used in modal logic literature.

The symbols \( I, C \) come from *algebraic* and *topological* interpretation of modal logics.

\( I \) corresponds to the topological *interior* of the set and \( C \) to its *closure*. 
The idea of a modal logic was first formulated by an American philosopher, C.I. Lewis in 1918.

Lewis has proposed yet another interpretation of lasting consequences, of the logical implication.

He created a notion of a modal truth, which lead to the notion of modal logic.

He did it in an attempt to avoid, what some felt, the paradoxes of semantics for classical implication which accepts as true that a false sentence implies any sentence.
Introduction to Modal Logics

Lewis’ notions appeal to epistemic considerations and the whole area of modal logics bristles with philosophical difficulties and hence the numbers of modal logics have been created.

Unlike the classical connectives, the modal connectives do not admit of truth-functional interpretation, i.e. do not accept the extensional semantics.

This was the reason for which modal logics were first developed as proof systems, with intuitive notion of semantics expressed by the set of adopted axioms.
Introduction to Modal Logics

The first definition of modal semantics, and hence the proofs of the completeness theorems came some 20 years later.

It took yet another 25 years for discovery and development of the second and more general approach to the modal semantics.

These are the two established ways of interpret modal connectives, i.e. to define the modal semantics.
Introduction to Modal Logics

The historically, the first modal semantics is due to Mc Kinsey and Tarski (1944, 1946)
It is a topological semantics that provides a powerful mathematical interpretation of some of modal logics, namely modal S4 and S5

It connects the modal notion of necessity with the topological notion of the interior of a set, and the modal notion of possibility with the notion of the closure of a set

Our choice of symbols $\Box$ and $\Diamond$ for necessity and possibility connectives comes from this interpretation

The topological interpretation mathematically powerful as it is, is less universal in providing models for other modal logics
The most recent and the most general semantics is due to Kripke (1964). It uses the notion of possible worlds.

Roughly, we say that the formula $CA$ is true if $A$ is true in some possible world, called actual world.

The formula $IA$ is true if $A$ is true in every possible world.

We present here a short version of the topological semantics in a form of algebraic models.

We leave the Kripke semantics for the reader to explore from other, multiple sources.
Introduction to Modal Logics

As we have already mentioned, modal logics were first developed, as was the intuitionistic logic, in a form of proof systems only.

First Hilbert style modal proof system was published by Lewis and Langford in 1932.

They presented a formalization for two modal logics, which they called $S_1$ and $S_2$.

They also outlined three other proof systems, called $S_3$, $S_4$, and $S_5$. 
Introduction to Modal Logics

Since then hundreds of modal logics have been created. There are some standard books in the subject.

These are, between the others:
Hughes and Cresswell (1969) for philosophical motivation for various modal logics and intuitionistic logic,

Bowen (1979) for a detailed and uniform study of Kripke models for modal logics,

Segeberg (1971) for excellent modal logics classification,
Fitting (1983), for extended and uniform studies of automated proof methods for classes of modal logics.
Hilbert Style Modal Proof Systems
Hilbert Style Modal Proof Systems

We present now Hilbert style formalization for S4 and S5 logics due to Mc Kinsey and Tarski (1948) and Rasiowa and Sikorski (1964)

We also discuss the relationship between S4 and S5, and between the intuitionistic logic and S4 modal logic, as first observed by Gödel

The formalizations stress the connection between S4, S5 and topological spaces which constitute models for them
Modal Language

Modal Language
We add two extra one argument connectives \( I \) and \( C \) to the propositional language \( \mathcal{L}_{\{\cup, \cap, \Rightarrow, \neg\}} \), i.e. we adopt

\[
\mathcal{L} = \mathcal{L}_{\{\cup, \cap, \Rightarrow, \neg, I, C\}}
\]

as the modal language. We read a formulas \( IA, CA \) as necessary A and possible A, respectively.

The language \( \mathcal{L}_{\{\cup, \cap, \Rightarrow, \neg, I, C\}} \) is common to all modal logics.

Modal logics differ on a choice of axioms and rules of inference, when studied as proof systems and on a choice of respective semantics.
McKinsey, Tarski Proof Systems

As modal logics extend the classical logic, any modal logic contains two groups of axioms: classical and modal

McKinsey, Tarski (1948)

AG1 classical axioms

We adopt as classical axioms any complete set of axioms under classical semantics

AG2 modal axioms

M1 (IA ⇒ A)
M2 (I(A ⇒ B) ⇒ (IA ⇒ IB))
M3 (IA ⇒ IIA)
M4 (CA ⇒ ICA)
Modal S4 and S5

Rules of inference

\[(MP) \quad \frac{A \; (A \Rightarrow B)}{B}, \quad \text{and} \quad (I) \quad \frac{A}{IA} \]

The modal rule \((I)\) was introduced by Gödel and is referred to as a necessitation rule.

We define modal proof systems \(S4\) and \(S5\) as follows

\[ S4 = (L, F, \text{classical axioms, } M1 - M3, (MP), (I) ) \]
\[ S5 = (L, F, \text{classical axioms, } M1 - M4, (MP), (I) ) \]
Modal S4 and S5

Observe that the axioms of S5 extend the axioms of S4 and both systems share the same inference rules, hence we have immediately the following

Fact For any formula $A \in \mathcal{F}$,

if $\vdash_{S4} A$, then $\vdash_{S5} A$
It is often the case, as it is for S4 and S5, that modal connectives are definable by each other. We define them as follows:

\[ IA = \neg C \neg A, \quad \text{and} \quad CA = \neg I \neg A \]

Language
We hence assume now that the language \( \mathcal{L} \) of Rasiowa, Sikorski modal proof systems contains only one modal connective. We choose it to be \( I \) and adopt the following language:

\[ \mathcal{L} = \mathcal{L}\{\cap, \cup, \Rightarrow, \neg, I\} \]

There are, as before, two groups of axioms: classical and modal.
Rasiowa, Sikorski (1964)

AG1 classical axioms
We adopt as classical axioms any complete set of axioms under classical semantics

AG2 modal axioms
R1 \(((\mathbf{I}A \cap \mathbf{I}B) \Rightarrow \mathbf{I}(A \cap B))\)
R2 \((\mathbf{I}A \Rightarrow A)\)
R3 \((\mathbf{I}A \Rightarrow \mathbf{II}A)\)
R4 \(\mathbf{I}(A \cup \neg A)\)
R5 \((\neg \mathbf{I}\neg A \Rightarrow \mathbf{I}\neg \mathbf{I}\neg A)\)
Modal RS4 and RS5

Rules of inference

We adopt the Modus Ponens and an additional rule $(RI)$

\[
(MP) \quad \frac{A \land (A \Rightarrow B)}{B} \quad \text{and} \quad (RI) \quad \frac{(A \Rightarrow B)}{(IA \Rightarrow IB)}
\]

We define modal proof systems $RS4$ and $RS5$ as follows

\[
RS4 = (\mathcal{L}, \mathcal{F}, \text{classical axioms, } R1 - R4, (MP), (RI))
\]

\[
RS5 = (\mathcal{L}, \mathcal{F}, \text{classical axioms, } R1 - R5, (MP), (RI))
\]
Modal RS4 and RS5

Observe that the axioms of RS5 extend the axioms of RS4 and both systems share the same inference rules, hence we have immediately the following

Fact  For any formula $A \in \mathcal{F}$,

\[
\text{if } \vdash_{RS4} A, \text{ then } \vdash_{RS5} A
\]
Algebraic Semantics for S4 and S5
Algebraic Semantics for S4 and S5

The McKinsey, Tarski proof systems S4, S5 and Rasiowa, Sikorski proof systems RS4, RS5 are complete with the respect to both topological semantics, and Kripke semantics.

We shortly discuss the topological semantics, and algebraic completeness theorems.

We leave the Kripke semantics for the reader to explore from other, multiple sources.
Algebraic Semantics for S4 and S5

The **topological semantics** was initiated by McKinsey and Tarski in 1946, 1948 and consequently developed into a field of **Algebraic Logic**

The **algebraic** approach to logic is presented in detail in now classic algebraic logic books:

”Mathematics of Metamathematics”, Rasiowa, Sikorski (1964),


We want to point out that the **first idea** of a connection between **modal** propositional logic and **topology** is due to Tang Tsao -Chen, (1938) and Dugunji (1940)
Here are some basic definitions

**Boolean Algebra**

An abstract algebra $B = (B, 1, 0, \Rightarrow, \cap, \cup, \neg)$ is said to be a **Boolean algebra** if it is a distributive lattice and every element $a \in B$ has a complement $\neg a \in B$

**Topological Boolean algebra**

By a topological Boolean algebra we mean an abstract algebra

$$B = (B, 1, 0, \Rightarrow, \cap, \cup, \neg, I)$$

where $(B, 1, 0, \Rightarrow, \cap, \cup, \neg)$ is a **Boolean algebra** and, moreover, the following conditions hold for any $a, b \in B$

$$l(a \cap b) = la \cap lb, \quad la \cap a = la, \quad lla = la, \quad \text{and} \quad l1 = 1$$
Algebraic Semantics for S4 and S5

The element $Ia$ is called a **interior** of $a$
The element $\neg I \neg a$ is called a **closure** of $a$ and will be **denoted** by $Ca$
Thus the operations $I$ and $C$ are such that

$$Ca = \neg I \neg a \quad \text{and} \quad Ia = \neg C \neg a$$

In this case we write the **topological Boolean algebra** as

$$\mathcal{B} = (B, 1, 0, \Rightarrow, \cap, \cup, \neg, I, C)$$

It is easy to prove that in in any topological Boolean algebra the following **conditions** hold for any $a, b \in B$

$$C(a \cup b) = Ca \cup Cb, \quad Ca \cup a = Ca, \quad CCa = Ca \quad \text{and} \quad C0 = 0$$
Example
Let $X$ be a topological space with an interior operation $I$
Then the family $\mathcal{P}(X)$ of all subsets of $X$ is a topological Boolean algebra with $1 = X$, with
the operation $\Rightarrow$ defined by the formula

$$Y \Rightarrow Z = (X - Y) \cup Z$$

for all subsets $Y, Z$ of $X$
and with set-theoretical operations of union, intersection, complementation, and the interior operation $I$

Every sub algebra of this algebra is a topological Boolean algebra, called a topological field of sets or, more precisely, a topological field of subsets of $X$
Algebraic Semantics for S4 and S5

Given a topological Boolean algebra

\[(B, 1, 0, \Rightarrow, \cap, \cup, \neg)\]

The element \(a \in B\) is said to be open (closed) if \(a = Ia\) (\(a = Ca\))

**Clopen Topological Boolean Algebra**

A topological Boolean algebra

\[\mathcal{B} = (B, 1, 0, \Rightarrow, \cap, \cup, \neg, I, C)\]

such that every open element is closed and every closed element is open, i.e. such that for any \(a \in B\)

\[Cl a = Ia\quad \text{and} \quad IC a = Ca\]

is called a clopen topological Boolean algebra
S4, S5 Tautology

We loosely say that a formula $A$ is a modal $S4$ tautology if and only if any topological Boolean algebra is a model for $A$.

We say that $A$ is a modal $S5$ tautology if and only if any clopen topological Boolean algebra is a model for $A$.

We put it formally as follows.
Modal Algebraic Model

For any formula $A$ of a modal language $\mathcal{L}_{\{\cup, \cap, \Rightarrow, \neg, I, C\}}$ and for any topological Boolean algebra $B = (B, 1, 0, \Rightarrow, \cap, \cup, \neg, I, C)$, the algebra $B$ is a model for the formula $A$ and denote it by $B \models A$ if and only if $v^*(A) = 1$ holds for all variables assignments $v : VAR \rightarrow B$. 
S4, S5 Tautology

**Definition of S4 Tautology**
A formula \( A \) is a modal S4 \textbf{tautology} and is denoted by
\[
\models_{S4} A
\]
if and only if for all topological Boolean algebras \( B \) we have that
\[
B \models A
\]

**Definition of S5 Tautology**
A formula \( A \) is a modal S5 \textbf{tautology} and is denoted by
\[
\models_{S5} A
\]
if and only if for all clopen topological Boolean algebras \( B \) we have that
\[
B \models A
\]
S4, S5 Completeness Theorem

We write $\vdash_{S4} A$ and $\vdash_{S5} A$ do denote provability any proof system for modal S4, S5 logics and in particular the proof systems defined here.

Completeness Theorem

For any formula $A$ of the modal language $\mathcal{L}\{\cup, \cap, \Rightarrow, \neg, I, C\}$

$\vdash_{S4} A$ if and only if $\models_{S4} A$

$\vdash_{S5} A$ if and only if $\models_{S5} A$

The completeness for S4, S4 follows directly from the following general Algebraic Completeness Theorems.
S4 Algebraic Completeness Theorem

For any formula $A$ of the modal language $\mathcal{L}\{\cup, \cap, \Rightarrow, \neg, I, C\}$ the following conditions are equivalent

(i) $\vdash_{S4} A$
(ii) $\models_{S4} A$
(iii) $A$ is valid in every topological field of sets $\mathcal{B}(X)$
(iv) $A$ is valid in every topological Boolean algebra $\mathcal{B}$ with at most $2^{2^r}$ elements, where $r$ is the number of all subformulas of $A$

(iv) $v^*(A) = X$ for every variable assignment $v$ in the topological field of sets $\mathcal{B}(X)$ of all subsets of a dense-in-itself metric space $X \neq \emptyset$ (in particular of an n-dimensional Euclidean space $X$)
S4 Algebraic Completeness Theorem

S5 Algebraic Completeness Theorem
For any formula $A$ of the modal language $L\{\cup, \cap, \Rightarrow, \neg, I, c\}$ the following conditions are equivalent

(i) $\vdash S_5 A$

(ii) $|= S_5 A$

(iii) $A$ is valid in every clopen topological field of sets $\mathcal{B}(X)$

(iv) $A$ is valid in every clopen topological Boolean algebra $\mathcal{B}$ with at most $2^{2^r}$ elements, where $r$ is the number of all sub formulas of $A$
S4 and S5 Decidability

The equivalence of conditions (i) and (iv) of the Algebraic Completeness Theorems proves the **semantical** decidability of modal S4 and S5

**S4, S5 Decidability**

Any complete S4, S5 proof system is **semantically decidable**, i.e. the following holds

\[ \vdash_{S4} A \quad \text{if and only if} \quad B \models A \]

for every topological Boolean algebra \( B \) with at most \( 2^{2^r} \) elements, where \( r \) is the number of all sub formulas of \( A \)

Similarly, we also have

\[ \vdash_{S5} A \quad \text{if and only if} \quad B \models A \]

for every **clopen** topological Boolean algebra \( B \) with at most \( 2^{2^r} \) elements, where \( r \) is the number of all sub formulas of \( A \)
S4 and S5 Syntactic Decidability

**S4, S5 Syntactic Decidability** (Wasilewska 1967, 1971)

Rasiowa stated in 1950 an **open problem** of providing a cut-free RS type formalization for modal propositional S4 calculus.

Wasilewska solved this open problem in 1967 and presented the result at the ASL Summer School and Colloquium in Mathematical Logic, Manchester, August 1969.

It appeared in print as *A Formalization of the Modal Propositional S4-Calculus*, Studia Logica, North Holland, XXVII (1971).
S4 and S5 Syntactic Decidability

The paper also contained an algebraic proof of completeness theorem followed by Gentzen cut-elimination theorem, the Hauptzatz

The resulting implementation written in LISP-ALGOL was the first modal logic theorem prover created. It was done with collaboration with B. Waligorski and the authors didn’t think it to be worth a separate publication. Its existence was only mentioned in the published paper.

The S5 Syntactic Decidability follows from the one for S4 and the following Embedding Theorems.
Modal S4 and Modal S5

The relationship between $S4$ and $S5$ was first established by Ohnishi and Matsumoto in 1957-59 and is as follows.

**Embedding 1**
For any formula $A \in \mathcal{F}$,

$\models_{S4} A$ if and only if $\models_{S5} ICA$

$\vdash_{S4} A$ if and only if $\vdash_{S5} ICA$

**Embedding 2**
For any formula $A \in \mathcal{F}$

$\models_{S5} A$ if and only if $\models_{S4} ICIA$

$\vdash_{S5} A$ if and only if $\vdash_{S4} ICIA$
On S4 derivable disjunction

In a classical logic it is possible for the disjunction \((A \cup B)\) to be a tautology when neither \(A\) nor \(B\) is a tautology. This does not hold for the intuitionistic logic. We have a following theorem similar to the intuitionistic case to the for modal S4

**Theorem** McKinsey, Tarski (1948)

A disjunction \((IA \cup IB)\) is S4 provable if and only if either \(A\) or \(B\) S4 provable, i.e.

\[
\vdash_{S4} (IA \cup IB) \quad \text{if and only if} \quad \vdash_{S4} A \quad \text{or} \quad \vdash_{S4} B
\]
S4 and Intuitionistic Logic, S5 and Classical Logic
S4 and Intuitionistic Logic

As we have said in the introduction, Gödel was the first to consider the connection between the intuitionistic logic and a logic which was named later S4.

Gödel’s proof was purely syntactic in its nature, as the semantics for neither intuitionistic logic nor modal logic S4 had not been invented yet.

The algebraic proof of this fact, was first published by McKinsey and Tarski in 1948.
S4 and Intuitionistic Logic

We define here the Gödel-Tarski mapping establishing the S4 and intuitionistic logic connection.

We refer the reader to Rasiowa, Sikorski book “Mathematics of Metamathematics” (1965) for the algebraic proofs of its properties and respective theorems.
S4 and Intuitionistic Logic

Let $\mathcal{L}$ be a propositional language of modal logic i.e the language

$$\mathcal{L} = \mathcal{L}\{\cap, \cup, \Rightarrow, \neg, \bot\}$$

Let $\mathcal{L}_0$ be a language obtained from $\mathcal{L}$ by elimination of the connective $\bot$ and by the replacement the classical negation connective $\neg$ by the intuitionistic negation, which we will denote here by a symbol $\sim$.

Such obtained language

$$\mathcal{L}_0 = \mathcal{L}\{\cap, \cup, \Rightarrow, \sim\}$$

is a propositional language of the intuitionistic logic.
S4 and Intuitionistic Logic

In order to establish the connection between the languages $\mathcal{L}$ and $\mathcal{L}_0$ and hence between modal and intuitionistic logic, we consider a mapping $f$ which to every formula $A \in F_0$ of $\mathcal{L}_0$ assigns a formula $f(A) \in \mathcal{F}$ of $\mathcal{L}$

We define the mapping $f$ as follows
Gödel - Tarski Mapping

Definition of Gödel-Tarski mapping
A function

\[ f: \mathcal{F}_0 \rightarrow \mathcal{F} \]

such that

\[ f(a) = Ia \quad \text{for any} \quad a \in \text{VAR} \]

\[ f((A \Rightarrow B)) = I(f(A) \Rightarrow f(B)) \]

\[ f((A \cup B)) = (f(A) \cup f(B)) \]

\[ f((A \cap B)) = (f(A) \cap f(B)) \]

\[ f(\sim A) = I\neg f(A) \]

where \( A, B \) are any formulas in \( \mathcal{L}_0 \) is called a Gödel-Tarski mapping.
Example

Let $A$ be a formula

$$(((\neg A \cap \neg B) \Rightarrow \neg (A \cup B)))$$

and $f$ be the Gödel-Tarski mapping. We evaluate $f(A)$ as follows

$$f(((\neg A \cap \neg B) \Rightarrow \neg (A \cup B))) =$$

$I(f(\neg A \cap \neg B) \Rightarrow f(\neg (A \cup B))) =$

$I((f(\neg A) \cap f(\neg B)) \Rightarrow f(\neg (A \cup B))) =$

$I(\neg fA \cap \neg fB) \Rightarrow \neg f(A \cup B)) =$

$I((\neg A \cap \neg B) \Rightarrow \neg (fA \cup fB)) =$

$I((\neg A \cap \neg B) \Rightarrow \neg (A \cup B))$
The following theorem established relationship between intuitionistic and modal S4 logics

**Theorem**
Let $f$ be the Gödel-Tarski mapping
For any formula $A$ of intuitionistic language $\mathcal{L}_0$, 

$$\vdash_I A \text{ if and only if } \vdash_{S4} f(A)$$

where $I, S4$ denote any proof systems for intuitionistic and S4 logic, respectively
Classical Logic and Modal S5

In order to establish the connection between the modal S5 and classical logics we consider the following Gödel-Tarski mapping between the modal language $\mathcal{L}_{\{\cap, \cup, \Rightarrow, \neg, \mathbf{I}\}}$ and its classical sub-language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$

With every classical formula $A$ we associate a modal formula $g(A)$ defined by induction on the length of $A$ as follows:

$$g(a) = \mathbf{I}a,$$
$$g((A \Rightarrow B)) = \mathbf{I}(g(A) \Rightarrow g(B)),$$
$$g((A \cup B)) = (g(A) \cup g(B)),$$
$$g((A \cap B)) = (g(A) \cap g(B)),$$
$$g(\neg A) = \mathbf{I}\neg g(A)$$
Classical Logic and Modal S5

The following theorem establishes relationship between classical and S5 logics

**Theorem**
Let $g$ be the Gödel-Tarski mapping between $L\{\neg, \cap, \cup, \Rightarrow\}$ and $L\{\cap, \cup, \Rightarrow, \neg, \top\}$

For any formula $A$ of $L\{\neg, \cap, \cup, \Rightarrow\}$,

$$\vdash_H A \quad \text{if and only if} \quad \vdash_{S5} g(A)$$

where $H, S5$ denote any proof systems for classical and and S5 modal logic, respectively