

CSE581

Computer Science Fundamentals: Theory

Professor Anita Wasilewska

P1 LOGIC: LECTURE 7

Chapter 7

Introduction to Intuitionistic and Modal Logics

PART 1: Intuitionistic Logic: Philosophical Motivation

Intuitionistic Logic: Philosophical Motivation

Intuitionistic logic has developed as a result of certain philosophical views on the foundation of mathematics, known as **intuitionism**

Intuitionism was originated by **L. E. J. Brouwer** in **1908**

The first **Hilbert style** formalization of the **intuitionistic logic**, formulated as a proof system, is due to **A. Heyting (1930)**

We **present** a **Hilbert** style proof system / that is equivalent to the **Heyting's** original formalization

We also **discuss** the **relationship** between intuitionistic and classical logic.

Intuitionistic Logic: Philosophical Motivation

There have been several **successful** attempts at creating **semantics** for the **intuitionistic logic**. The most **recent** called **Kripke models** were defined by **Kripke** in **1964**

The **first intuitionistic** semantics was defined in a form of **pseudo-Boolean** algebras by **McKinsey** and **Tarski** in years **1944 - 1946**

Their **algebraic** approach to intuitionistic and classical semantics was followed by many **authors** and developed into a **new field** of **Algebraic Logic**

The **pseudo- Boolean** algebras are called also **Heyting** algebras to memorize his **first** accepted formalization of the **intuitionistic** logic as a proof system

Intuitionistic Logic: Philosophical Motivation

An uniform presentation of **algebraic models** for **classical**, **intuitionistic** and **modal logics S4, S5** was first given in a now classic **algebraic logic** book:

"Mathematics of Metamathematics", **Rasiowa, Sikorski (1964)**

The main **goal** of this chapter is to give a presentation of the **intuitionistic logic** formulated as **Hilbert** and **Gentzen** proof systems

We also **discuss** its **algebraic** semantics and the fundamental theorems that establish the **relationship** between **classical** and **intuitionistic** propositional logics

Intuitionistic Logic: Philosophical Motivation

Intuitionists' **view-point** on the **meaning** of the basic logical and set theoretical **concepts** used in mathematics **is different** from that of most **mathematicians** use in their research

The basic **difference** between the **intuitionist** and **classical** mathematician lies in the **interpretation** of the word **exists**. For example, let $A(x)$ be a statement in the **arithmetic** of natural numbers. For the **mathematicians** the sentence $\exists x A(x)$ is **true** if it is a **theorem** of arithmetic

If a **mathematician** **proves** sentence $\exists x A(x)$ this **does not always** mean that he is able to indicate a **method of construction** of a natural number n such that $A(n)$ holds

Intuitionistic Logic: Philosophical Motivation

Moreover, the **mathematician** often obtains the **proof** of the existential sentence $\exists x A(x)$ by **proving** first a sentence

$$\neg \forall x \neg A(x)$$

Next he makes use of a **classical** tautology

$$(\neg \forall x \neg A(x)) \Rightarrow \exists x A(x)$$

By applying **Modus Ponens** he obtains the **proof** of the existential sentence

$$\exists x A(x)$$

For the **intuitionist** such method is **not acceptable**, for it **does not** give any method of **constructing** a number n such that $A(n)$ holds

Intuitionistic Logic: Philosophical Motivation

For this reason the **intuitionist** **do not accept** the **classical** tautology

$$(\neg \forall x \neg A(x)) \Rightarrow \exists x A(x)$$

as **intuitionistic** tautology, or as an **intuitionistically provable** sentence

Intuitionistic Logic: Philosophical Motivation

Let us denote by $\vdash_I A$ and $\models_I A$ the fact that A is intuitionistically provable and that A is intuitionistic tautology, respectively

The **proof system** I for the intuitionistic logic has hence to be such that

$$\not\vdash_I (\neg\forall x \neg A(x)) \Rightarrow \exists x A(x))$$

The intuitionistic semantics I has to be such that

$$\not\models_I (\neg\forall x \neg A(x)) \Rightarrow \exists x A(x))$$

Intuitionistic Logic: Philosophical Motivation

The above means also that **intuitionists** interpret **differently** the meaning of propositional **connectives**

Intuitionistic implication

The **intuitionistic** implication $(A \Rightarrow B)$ is considered by to be **true** if there **exists** a method by which a proof of **B** can be **deduced** from the proof of **A**

In the case of the implication

$$(\neg \forall x \neg A(x)) \Rightarrow \exists x A(x)$$

there is no general method which, from a proof of the sentence

$$(\neg \forall x \neg A(x))$$

permits us to **obtain** an **intuitionistic proof** of the sentence

$$\exists x A(x)$$

Intuitionistic Logic: Philosophical Motivation

Intuitionistic negation

The sentence $\neg A$ is considered **intuitionistically true** if the **acceptance** of the sentence A leads to **absurdity**

As a result of above understanding of **negation** and **implication** we have that in the **intuitionistic** proof system I

$$\vdash_I (A \Rightarrow \neg\neg A) \quad \text{but} \quad \not\vdash_I (\neg\neg A \Rightarrow A)$$

Consequently, the **intuitionistic** semantics I has to be such that

$$\models_I (A \Rightarrow \neg\neg A) \quad \text{and} \quad \not\models_I (\neg\neg A \Rightarrow A)$$

Intuitionistic Logic: Philosophical Motivation

Intuitionistic disjunction

The **intuitionist** regards a **disjunction** $(A \cup B)$ as **true** if **one** of the sentences A, B is **true** and **there is** a method by which it is possible to find out **which** of them is **true**

As a consequence a **classical** law of **excluded middle**

$$(A \cup \neg A)$$

is not acceptable by the **intuitionists**

This means that the **intuitionistic** proof system I must be such that

$$\not\vdash_I (A \cup \neg A)$$

and the **intuitionistic** semantics I has to be such that

$$\not\models_I (A \cup \neg A)$$

Chapter 7

Introduction to Intuitionistic and Modal Logics

PART 2: Intuitionistic Proof System *I*,
Algebraic Semantics and Completeness Theorem

Intuitionistic Proof System /

We define now a **Hilbert** style **proof system** / with a set of axioms that is due to **Rasiowa (1959)**. We adopted this **axiomatization** for two reasons

First reason is that it is the **most natural** and **appropriate** set of axioms to carry the the **algebraic proof** of the **completeness theorem**

Second reason is that they clearly describe the main **difference** between **intuitionistic** and **classical** logic
Namely, by **adding** to / the only **one** more axiom

$$(A \cup \neg A)$$

we get a **complete** formalization for **classical** logic

Intuitionistic Proof System /

Here are the components if the proof system /

Language

We adopt a propositional language

$$\mathcal{L} = \mathcal{L}_{\{\cup, \cap, \Rightarrow, \neg\}}$$

with the set of formulas \mathcal{F}

Axioms

$$\text{A1} \quad ((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$$

$$\text{A2} \quad (A \Rightarrow (A \cup B))$$

$$\text{A3} \quad (B \Rightarrow (A \cup B))$$

$$\text{A4} \quad ((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$$

$$\text{A5} \quad ((A \cap B) \Rightarrow A)$$

$$\text{A6} \quad ((A \cap B) \Rightarrow B)$$

$$\text{A7} \quad ((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B))))$$

Intuitionistic Proof System I

A7 $((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \wedge B))))$

A8 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \wedge B) \Rightarrow C))$

A9 $((((A \wedge B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))),$

A10 $(A \wedge \neg A) \Rightarrow B),$

A11 $((A \Rightarrow (A \wedge \neg A)) \Rightarrow \neg A),$

where A, B, C are any formulas in \mathcal{L}

Rules of inference

We adopt the **Modus Ponens**

$$(MP) \frac{A ; (A \Rightarrow B)}{B}$$

as the **only** rule of inference

Intuitionistic Proof System I

A proof system

$$I = (\mathcal{L}, \mathcal{F} \text{ A1} - \text{A11}, (MP))$$

for axioms **A1 - A11** defined above is called a **Hilbert** style **formalization** for **intuitionistic** propositional logic

We introduce, as usual, the notion of a **formal proof** in I and denote by

$$\vdash_I A$$

the fact that a formula **A** has a formal **proof** in I or that **A** is **provable** in I

Algebraic Semantics and Completeness Theorem

Algebraic Semantics

We present now a short version of **Tarski**, **Rasiowa**, and **Sikorski** **pseudo-Boolean** algebra semantics

We also discuss the **algebraic completeness theorem** for the **intuitionistic** propositional logic

We leave the **Kripke semantics** for the reader to **explore** from other, multiple **sources**

Algebraic Semantics

Here are some **basic** definitions

Relatively Pseudo-Complemented Lattice (Birkhoff, 1935)

A lattice

$$(B, \cap, \cup)$$

is said to be relatively pseudo-complemented if and only if for any elements $a, b \in B$, there exists the **greatest** element c , such that

$$a \cap c \leq b$$

Such greatest element c is denoted by $a \Rightarrow b$ and called the **pseudo-complement** of a **relative** to b

Algebraic Semantics

Directly from definition we have that

(*) $x \leq a \Rightarrow b$ if and only if $a \cap x \leq b$ for all $x, a, b \in B$

This equation (*) can serve as the **definition** of the relative pseudo-complement $a \Rightarrow b$

Algebraic Semantics

Fact

Every relatively pseudo-complemented lattice (B, \cap, \cup) has the **greatest** element, called a **unit element** and denoted by **1**

Proof

Observe that $a \cap x \leq a$ for all $x, a \in B$

By $(*)$ we have that $x \leq a \Rightarrow a$ for all $x \in B$

This means that $a \Rightarrow a$ is the greatest element in the lattice (B, \cap, \cup) . We write it as

$$a \Rightarrow a = 1$$

Algebraic Semantics

Definition

An abstract algebra

$$\mathcal{B} = (B, 1, \Rightarrow, \cap, \cup)$$

is said to be a **relatively pseudo-complemented lattice** if and only if (B, \cap, \cup) is a relatively pseudo-complemented lattice with the relative pseudo-complement \Rightarrow defined by the equation

$$(*) \quad x \leq a \Rightarrow b \quad \text{if and only if} \quad a \cap x \leq b \quad \text{for all} \quad x, a, b \in B$$

and with the **unit** element **1**

Algebraic Semantics

Relatively Pseudo-complemented Set Lattices

Consider a **topological** space X with an interior operation I

Let $\mathcal{G}(X)$ be the class of all **open** subsets of X and

$\mathcal{G}^*(X)$ be the class of all both **dense** and **open** subsets of X

Then the algebras

$$(\mathcal{G}(X), X, \cup, \cap, \Rightarrow), \quad (\mathcal{G}^*(X), X, \cup, \cap, \Rightarrow)$$

where \cup, \cap are set-theoretical operations of **union**,
intersection, and \Rightarrow is defined by

$$Y \Rightarrow Z = I(X - Y) \cup Z$$

are relatively pseudo-complemented lattices

Algebraic Semantics

Clearly, all **sub algebras** of these algebras are also **relatively pseudo-complemented lattices**

They are typical **examples** of relatively pseudo complemented lattices

Algebraic Semantics

Pseudo - Boolean Algebra (Heyting Algebra)

An algebra

$$\mathcal{B} = (B, 1, 0, \Rightarrow, \cap, \cup, \neg)$$

is said to be a **pseudo - Boolean** algebra if and only if

$$(B, 1, \Rightarrow, \cap, \cup)$$

is a relatively pseudo-complemented **lattice** in which a **zero** element **0** exists and \neg is a one argument **operation** defined as follows

$$\neg a = a \Rightarrow 0$$

Algebraic Semantics

The operation \neg defined as

$$\neg a = a \Rightarrow 0$$

is called a **pseudo-complementation**

The **pseudo - Boolean** algebras are also called **Heyting** algebras to stress their connection to the **intuitionistic** logic

Algebraic Semantics

Let X be **topological** space with an **interior** operation I

Let $\mathcal{G}(X)$ be the class of all **open** subsets of X

Then

$$(\mathcal{G}(X), X, \emptyset, \cup, \cap, \Rightarrow, \neg)$$

where \cup, \cap are set-theoretical operations of **union**, **intersection**, and \Rightarrow is defined by

$$Y \Rightarrow Z = I(X - Y) \cup Z$$

and \neg is defined as

$$\neg Y = Y \Rightarrow \emptyset = I(X - Y), \text{ for all } Y \subseteq X$$

is a **pseudo - Boolean** algebra

Every **sub algebra** of $\mathcal{G}(X)$ is also a pseudo-Boolean algebra

They are called **pseudo-fields of sets**

Algebraic Semantics

The following theorem states that **pseudo-fields** are typical **examples** of **pseudo - Boolean** algebras.

The theorems of this type are often called **Stone Representation Theorems** to remember an American mathematician **H. M. Stone**

Stone was one of the **first** to initiate the investigations of **relationship** between **logic** and general **topology** in the article

"The Theory of Representations for Boolean Algebras",
Trans. of the Amer.Math, Soc 40, **1936**

Algebraic Semantics

Representation Theorem (McKinsey, Tarski, 1946)

For every **pseudo - Boolean** algebra

$$\mathcal{B} = (B, 1, 0, \Rightarrow, \cap, \cup, \neg)$$

there exists a **monomorphism** h of \mathcal{B} into a **pseudo-field** $\mathcal{G}(X)$ of all **open** subsets of a **compact** topological T_0 space X

Intuitionistic Algebraic Model

We say that a formula A is an **intuitionistic tautology**

if and only if

any **pseudo-Boolean** algebra \mathcal{B} is a **model** for A

This kind of **models** because their **connection** to abstract algebras are called **algebraic models**

We put it formally as follows.

Intuitionistic Algebraic Model

Intuitionistic Algebraic Model

Let A be a formula of the language $\mathcal{L}_{\{\cup, \cap, \Rightarrow, \neg\}}$ and let

$$\mathcal{B} = (B, 1, 0, \Rightarrow, \cap, \cup, \neg)$$

be a **pseudo - Boolean** algebra

We say that the algebra \mathcal{B} is a **model** for the formula A and denote it by

$$\mathcal{B} \models A$$

if and only if $v^*(A) = 1$ holds for all variables assignments

$$v : \text{VAR} \longrightarrow B$$

Intuitionistic Tautology

Intuitionistic Tautology

The formula A is an **intuitionistic tautology** and is denoted by

$$\models_I A$$

if and only if

$$\mathcal{B} \models A \quad \text{for all pseudo-Boolean algebras } \mathcal{B}$$

Intuitionistic Tautology

In **Algebraic Logic** the notion of **tautology** is often defined using a notion

"a formula **A** is **valid** in an algebra ***B***"

It is formally defined as follows

Intuitionistic Tautology

Definition

A formula **A** is **valid** in a pseudo-Boolean algebra

$$\mathcal{B} = (B, 1, 0, \Rightarrow, \cap, \cup, \neg)$$

if and only if $v^*(A) = 1$ holds for all variables assignments
 $v : VAR \rightarrow B$

Directly from definitions we get the following Fact

Intuitionistic Tautology

Fact

For any formula A ,

$\models_I A$ if and only if A is **valid**
in all pseudo-Boolean algebras \mathcal{B}

The **Fact** is often used as an equivalent **definition** of the
intuitionistic tautology

Intuitionistic Completeness

We write now $\vdash_I A$ to denote **any** proof system for the **intuitionistic** propositional logic, and in **particular** the **Rasiowa (1959)** proof system we have defined

Intuitionistic Completeness Theorem (Mostowski 1948)

For any formula A of $\mathcal{L}_{\{ \vee, \wedge, \Rightarrow, \neg \}}$,

$$\vdash_I A \quad \text{if and only if} \quad \models_I A$$

The above Mostowski 's theorem follows **directly** from a more general **intuitionistic algebraic completeness theorem** that combines results of **Mostowski (1958)**, **Rasiowa (1951)** and **Rasiowa-Sikorski (1957)**

Algebraic Completeness

Intuitionistic Algebraic Completeness Theorem

For any formula A the following conditions are equivalent

- (i) $\vdash_I A$
- (ii) $\models_I A$
- (iii) A is **valid** in every pseudo-Boolean algebra

$$(\mathcal{G}(X), X, \emptyset, \cup, \cap, \Rightarrow, \neg)$$

of **open** subsets of any **topological** space X

- (iv) A is **valid** in every pseudo-Boolean algebra \mathcal{B} with at most 2^{2^r} elements, where r is the number of all **sub formulas** of A

Moreover, each of the conditions (i) - (iv) is equivalent to the following one.

- (v) A is **valid** in the pseudo-Boolean algebra $(\mathcal{G}(X), X, \emptyset, \cup, \cap, \Rightarrow, \neg)$ of **open** subsets of a **dense-in-itself** metric space $X \neq \emptyset$ (in particular of an **n-dimensional Euclidean** space X)

Chapter 7

Introduction to Intuitionistic and Modal Logics

PART 3: Intuitionistic Tautologies and Connection with Classical Tautologies

Intuitionistic Tautologies

Here are some important **basic classical** tautologies that **are also intuitionistic tautologies**

$$(A \Rightarrow A)$$

$$(A \Rightarrow (B \Rightarrow A))$$

$$(A \Rightarrow (B \Rightarrow (A \cap B)))$$

$$((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$$

$$(A \Rightarrow \neg\neg A)$$

$$\neg(A \cap \neg A)$$

$$((\neg A \cup B) \Rightarrow (A \Rightarrow B))$$

Of course, all of logical axioms **A1 - A11** of the proof system **I** are also **classical** and **intuitionistic** tautologies

Intuitionistic Tautologies

Here are some **more** of important **classical** tautologies that **are also intuitionistic tautologies**

$$(\neg(A \cup B) \Rightarrow (\neg A \cap \neg B)) \quad \text{and} \quad ((\neg A \cap \neg B) \Rightarrow (\neg(A \cup B)))$$

$$((\neg A \cup \neg B) \Rightarrow \neg(A \cap B))$$

$$((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$$

$$((A \Rightarrow \neg B) \Rightarrow (B \Rightarrow \neg A))$$

$$(\neg\neg\neg A \Rightarrow \neg A)$$

$$(\neg A \Rightarrow \neg\neg\neg A)$$

$$(\neg\neg(A \Rightarrow B) \Rightarrow (A \Rightarrow \neg\neg B))$$

$$((C \Rightarrow A) \Rightarrow ((C \Rightarrow (A \Rightarrow B)) \Rightarrow (C \Rightarrow B)))$$

Intuitionistic Tautologies

Here are some important **classical** tautologies that **are not intuitionistic tautologies**

$$(A \cup \neg A)$$

$$(\neg\neg A \Rightarrow A)$$

$$((A \Rightarrow B) \Rightarrow (\neg A \cup B))$$

$$(\neg(A \cap B) \Rightarrow (\neg A \cup \neg B))$$

$$((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow A))$$

$$((\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A))$$

$$((A \Rightarrow B) \Rightarrow A) \Rightarrow A$$

Connection Between Classical and Intuitionistic Logics

Connection Between Classical and Intuitionistic Logics

The first **connection** is quite obvious

It was proved by **Rasiowa, Sikorski** in **1964** that by adding the axiom

$$\mathbf{A12} \quad (A \cup \neg A)$$

to the set of logical axioms **A1 - A11** of the proof system **I** we obtain a proof system **C** that is **complete** with respect to **classical** semantics

This proves the following

Theorem 1

Every formula that is **intuitionistically** derivable is also **classically** derivable, i.e. the implication

$$\text{If } \vdash_I A \text{ then } \vdash_C A$$

holds for any $A \in \mathcal{F}$

Classical and Intuitionistic Logics

We write $\models A$ and $\models_I A$ to denote that A is a **classical** and **intuitionistic** tautology, respectively.

As both proof systems **I** and **C** are **complete** under respective semantics, we can re-write **Theorem 1** as the following **relationship** between **classical** and **intuitionistic** tautologies

Theorem 2

For any formula $A \in \mathcal{F}$,

If $\models_I A$, then $\models A$

Classical and Intuitionistic Logics

The next **relationship** shows how to obtain **intuitionistic** tautologies from the **classical** tautologies and vice versa

The following has been proved by **Glivenko** in **1929** and independently by **Tarski** in **1938**

Theorem 3 (Glivenko, Tarski)

For any formula $A \in \mathcal{F}$,

A is classically provable if and only if $\neg\neg A$ is intuitionistically provable, i.e.

$$\vdash A \quad \text{if and only if} \quad \vdash_I \neg\neg A$$

where we use symbol \vdash for **classical** provability

Classical and Intuitionistic Logics

Theorem 4 (Tarski, 1938)

For any formula $A \in \mathcal{F}$,

A is a classical tautology if and only if $\neg\neg A$ is an intuitionistic tautology, i.e.

$$\models A \quad \text{if and only if} \quad \models_I \neg\neg A$$

Classical and Intuitionistic Logics

Theorem 5 (Gödel, 1931)

For any formulas $A, B \in \mathcal{F}$,

a formula $(A \Rightarrow \neg B)$ is classically provable if and only if it is intuitionistically provable, i.e.

$$\vdash (A \Rightarrow \neg B) \quad \text{if and only if} \quad \vdash_I (A \Rightarrow \neg B)$$

Classical and Intuitionistic Logics

Theorem 6 (Gödel, 1931)

For any formula $A, B \in \mathcal{F}$,

If A contains **no connectives** except \cap and \neg ,
then A is classically provable if and only if it is
intuitionistically provable, i.e

$\vdash A$ if and only if $\vdash_I A$

Classical and Intuitionistic Logics

By the **completeness** of classical and intuitionistic logics we get the following **semantic** version of **Gödel's Theorems 5, 6**

Theorem 7

A formula $(A \Rightarrow \neg B)$ is a **classical** tautology if and only if it is an **intuitionistic** tautology, i.e.

$$\models (A \Rightarrow \neg B) \quad \text{if and only if} \quad \models_I (A \Rightarrow \neg B)$$

Theorem 8

If a formula A contains no connectives except \cap and \neg , then

$$\models A \quad \text{if and only if} \quad \models_I A$$

On intuitionistically derivable disjunction

In **classical** logic it is possible for the disjunction

$$(A \cup B)$$

to be a **tautology** when neither **A** nor **B** is a **tautology**

The tautology $(A \cup \neg A)$ is the simplest example

This **does not hold** for the **intuitionistic** logic

This fact was **stated** without the proof by **Gödel** in **1931** and **proved** by **Gentzen** in **1935** via his proof system **LI** which was discussed shortly in **chapter 6** and is covered in detail in this chapter and the **next** set of slides

On intuitionistically derivable disjunction

The following theorem was announced without proof by Gödel in 1931 and proved by Gentzen in 1935

Theorem 9 (Gödel, Gentzen)

A disjunction $(A \cup B)$ is intuitionistically provable if and only if either A or B is intuitionistically provable i.e.

$$\vdash_I (A \cup B) \quad \text{if and only if} \quad \vdash_I A \quad \text{or} \quad \vdash_I B$$

On intuitionistically derivable disjunction

We obtain, via the **Completeness Theorems** the following **semantic** version of the above (Gödel, Gentzen) Theorem 9

Theorem 10

A disjunction $(A \cup B)$ is **intuitionistic** tautology if and only if either A or B is **intuitionistic** tautology, i.e.

$$\models_I (A \cup B) \text{ if and only if } \models_I A \text{ or } \models_I B$$