# CSE581
# Computer Science Fundamentals: Theory

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# P1 LOGIC: LECTURE 6

# Chapter 6
# Automated Proof Systems
# Completeness of Classical Propositional Logic

# Chapter 6
# Automated Proof Systems

## Lecture 6

PART 1: Proof System **RS**

Automated Search for Proofs: Decomposition Trees

PART 2: Proof System **RS**

Strong Soundness and Constructive Completeness

PART 3: Proof Systems **RS1, RS2**

# Chapter 6
# Automated Proof Systems

**Lecture 6a**

PART 4: Gentzen Sequent Systems **GL, G**

Strong Soundness and Constructive Completeness

**Lecture 6b**

PART 5: Original Gentzen Systems **LK, LI**

Classical and Intiutionistic Completeness and Hauptzatz Theorem

# Chapter 6
# Automated Proof Systems

## Lecture 6

PART 1: Proof System **RS**

Automated Search for Proofs: Decomposition Trees

# Gentzen Style Proof Systems

Hilbert style systems are easy to **define** and admit different proofs of **Completeness Theorem**

They are difficult to use by humans, not mentioning computer

Their emphasis is on logical **axioms**, keeping the **rules** of inference, with obligatory Modus Ponens, at a **minimum**

Gentzen style proof systems **reverse** this situation by emphasizing the importance of inference **rules**, reducing the role of logical **axioms** to an absolute **minimum**

# Gentzen Style Proof Systems

The Gentzen type systems may be less intuitive then the Hilbert systems but they allow us to **define** effective **automatic** procedures for proof search, what was **impossible** in a case of the Hilbert systems

For this reason they are called **automated proof systems**

They serve as formal models of **computing** systems that **automate** the reasoning process

# Gentzen Style Proof Systems

The Gentzen formalizations, as they are also called, were **invented** by **Gerald Gentzen** in 1934, hence the name

**Gentzen** proof systems for classical and intuitionistic **predicate** logics introduced special expressions built of formulas called **sequents**

This is why the Gentzen style systems using **sequents** as basic expressions are often called Gentzen sequent formalizations

# Gentzen Style Proof Systems

We present in **Lecture 6a** our own Gentzen sequent systems **GL** and **G** and prove their **completeness**

We also present a propositional version of Gentzen original system **LK** and discuss the original proof of his famous **Hauptsatz Theorem**

**Hauptsatz Theorem** is literally rendered as the Main Theorem and is known as a **Cut-elimination Theorem**

We prove the equivalency of the **cut-free** propositional **LK** and the **complete** proof system **G**

# Gentzen Style Proof Systems

A propositional version of Gentzen historical **original** formalization for intuitionistic logic **LI** is presented and discussed in Chapter 7

The original classical and intuitionistic **predicate** systems **LK** and **LI** are discussed in Chapter 9

## Gentzen Style Proof Systems

The other **historically important** automated proof systems **RS** and **QRS** are due to **Rasiowa** and **Sikorski** (1960)

Their proof systems for classical propositional and predicate logic use as basic expressions **sequences** of formulas, less complicated then Gentzen **sequents**

**Rasiowa** and **Sikorski** proof systems are simpler and easier to understand then the Gentzen **sequent** systems

This is one of the reasons the system **RS** is the first to be presented here

# Gentzen Style Proof Systems

**Historical** importance and lasting influence of **Rasiowa** and **Sikorski** work **lays** in the fact that they were the first to **use** the proof searching capacity of their proof systems to define a **constructive method** of proving the **completeness theorem** for both propositional and predicate classical logic

We introduce and explain in detail their **constructive method** and use it prove the **completeness** of the **RS** system and following systems **RS1** and **RS2**

# Gentzen Style Proof Systems

We also generalize the **RS** constructive method to the **Gentzen sequent** systems and prove the completeness of **GL** and **G**

The **completeness proof** for classical predicate system **RSQ** is presented in Chapter 9

# **RS** Proof System

# **RS** Proof System

**Components** of **RS**

**Language**

$$\mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$$

**Expressions**

We adopt as the set of expressions $\mathcal{E}$ the set $\mathcal{F}^*$ of all **finite sequences** of formulas

$$\mathcal{E} = \mathcal{F}^*$$

**Notation**

Elements of $\mathcal{E}$ are finite sequences of formulas and we denote them by

$$\Gamma, \Delta, \Sigma \ldots$$

with indices if necessary.

# **RS** Proof System

### Semantic Link

The the intuitive meaning of a sequence $\Gamma \in \mathcal{F}^*$ is that the truth assignment $v$ makes it **true** if and only if it makes the formula of the form of the **disjunction** of all formulas of $\Gamma$ **true**

For any sequence $\Gamma \in \mathcal{F}^*$

$$\Gamma = A_1, A_2, ..., A_n$$

we **denote**

$$\delta_\Gamma = A_1 \cup A_2 \cup ... \cup A_n$$

We define as the next step a **formal semantics** for **RS**

# Formal Semantics for **RS**

## Formal Semantics

Let $v : VAR \longrightarrow \{T, F\}$ be a truth assignment and $v^*$ its classical semantics **extension** to the set of formulas $\mathcal{F}$ We formally **extend** $v$ to the set $\mathcal{F}^*$ of all finite sequences of $\mathcal{F}$ as follows

$$v^*(\Gamma) = v^*(\delta_\Gamma) = v^*(A_1) \cup v^*(A_2) \cup ... \cup v^*(A_n)$$

# Formal Semantics for **RS**

**Model**

The sequence $\Gamma$ is said to be **satisfiable** if there is a truth assignment $v : VAR \longrightarrow \{T, F\}$ such that $v^*(\Gamma) = T$

We write it as

$$v \models \Gamma$$

and call $v$ a **model** for $\Gamma$

**Counter- Model**

The sequence $\Gamma$ is said to be **falsifiable** if there is a truth assignment $v$, such that $v^*(\Gamma) = F$

Such a truth assignment $v$ is called a **counter-model** for $\Gamma$

# Formal Semantics for **RS**

**Tautology**

The sequence $\Gamma$ is said to be a **tautology** if and only if $v^*(\Gamma) = T$ for all truth assignments $v : VAR \longrightarrow \{T, F\}$

We write

$$\models \; \Gamma$$

to denote that $\Gamma$ is a **tautology**

# Example

**Example**

Let $\Gamma$ be a sequence

$$a, (b \cap a), \neg b, (b \Rightarrow a)$$

The truth assignment $v$ such that

$$v(a) = F \quad \text{and} \quad v(b) = T$$

**falsifies** $\Gamma$, i.e. is a **counter-model** for $\Gamma$ as shows the following computation

$v^*(\Gamma) = v^*(\delta_\Gamma) = v^*(a) \cup v^*(b \cap a) \cup v^*(\neg b) \cup v^*(b \Rightarrow a) = F \cup (F \cap T) \cup F \cup (T \Rightarrow F) = F \cup F \cup F \cup F = F$

# Exercise

**Exercise**

**1.** Let $\Gamma$ be a sequence

$$a, (\neg b \cap a), \neg b, (a \cup b)$$

and let $v$ be a truth assignment for which $v(a) = T$

Prove that

$$v \models \Gamma$$

**2.** Let $\Gamma$ be a sequence

$$a, (\neg b \cap a), \neg b, (a \cup b)$$

Prove that

$$\models \Gamma$$

## Exercise

**Solution**

**1.** $\Gamma$ is a sequence

$$a, (\neg b \cap a), \neg b, (a \cup b)$$

We evaluate

$$v^*(\Gamma) = v^*(\delta_\Gamma) = v^*(a) \cup v^*(\neg b \cap a) \cup v^*(\neg b) \cup v^*(a \cup b) =$$
$$T \cup v^*(\neg b \cap a) \cup v^*(\neg b) \cup v^*(a \cup b) = T$$

We proved

$$v \models \Gamma$$

## Exercise

**Solution**

**2.** Assume now that $\Gamma$ is **falsifiable** i.e. that we have a truth assignment $v$ for which

$$v^*(\Gamma) = v^*(\delta_\Gamma) = v^*(a) \cup v^*(\neg b \cap a) \cup v^*(\neg b) \cup v^*(a \cup b) = F$$

This is possible **only when** (in short-hand notation)

$$a \cup (\neg b \cap a) \cup \neg b \cup a \cup b = F$$

what is **impossible** as $(\neg b \cup b) = T$ for all $v$

This **contradiction** proves that $\Gamma$ is a **tautology**

# Rules of inference

**Rules of inference** are of the form:

$$\frac{\Gamma_1}{\Gamma} \quad \text{or} \quad \frac{\Gamma_1 \ ; \ \Gamma_2}{\Gamma}$$

where $\Gamma_1, \Gamma_2$ are called **premisses** and $\Gamma$ is called the **conclusion** of the rule

Each rule of inference **introduces** a new logical connective or a negation of a logical connective

We **name** the rule that introduces the logical connective $\circ$ in the conclusion sequent $\Gamma$ by $(\circ)$

**The notation** $(\neg\circ)$ means that the negation of the logical connective $\circ$ is introduced in the conclusion sequence $\Gamma$

# Rules of inference of **RS**

**Rules of Inference**

**RS** contains seven inference rules:

$$(\cup), \quad (\neg\cup), \quad (\cap), \quad (\neg\cap), \quad (\Rightarrow), \quad (\neg \Rightarrow), \quad (\neg\neg)$$

Before we **define** the **rules** of **RS** we need to introduce some definitions.

# Literals

**Definition**

Any propositional variable, or a negation of propositional variable is called a **literal**

The set

$$LT = VAR \cup \{\neg a : \quad a \in VAR\}$$

is called a set of all propositional **literals**

The variables are called **positive literals**

Negations of variables are called **negative literals**

# Literals

We denote by

$$\Gamma^{'}, \quad \Delta^{'}, \quad \Sigma^{'} \ldots$$

finite sequences (empty included) formed out of **literals** i.e

$$\Gamma^{'}, \Delta^{'}, \Sigma^{'} \in LT^{*}$$

We will denote by

$$\Gamma, \quad \Delta, \quad \Sigma \ldots$$

the elements of $\mathcal{F}^{*}$

# Logical Axioms of **RS**

**Logical Axioms**

We adopt as an logical axiom of **RS** any sequence of **literals** which contains a propositional variable and its negation, i.e any sequence

$$\Gamma_1', \; a, \; \Gamma_2', \; \neg a, \; \Gamma_3'$$

$$\Gamma_1', \; \neg a, \; \Gamma_2', \; a, \; \Gamma_3'$$

where $a \in VAR$ is any **propositional variable**

We denote by LA the set of all logical axioms of **RS**

# Logical Axioms of **RS**

**Semantic Link**

Consider axiom

$$\Gamma'_1,\ a,\ \Gamma'_2,\ \neg a,\ \Gamma'_3$$

Directly from the extension of the notion of tautology to **RS** we have that for any truth assignment $v : VAR \longrightarrow \{T, F\}$

$v^*(\Gamma'_1, \neg a, \Gamma'_2, a, \Gamma'_3) = v^*(\Gamma'_1) \cup v^*(\neg a) \cup v^*(a) \cup v^*(\Gamma'_2, \Gamma'_3) =$
$v^*(\Gamma'_1) \cup T \cup v^*(\Gamma'_2, \Gamma'_3) = T$

The same applies to the axiom

$$\Gamma'_1,\ \neg a,\ \Gamma'_2,\ a,\ \Gamma'_3$$

We have thus proved the following.

**Fact** Logical axioms of **RS** are **tautologies**

## Inference Rules of **RS**

### Disjunction rules

$$(\cup)\quad \frac{\Gamma',\ A, B,\ \Delta}{\Gamma',\ (A \cup B),\ \Delta}, \qquad (\neg\cup)\quad \frac{\Gamma',\ \neg A,\ \Delta \quad ; \quad \Gamma',\ \neg B,\ \Delta}{\Gamma',\ \neg(A \cup B),\ \Delta}$$

### Conjunction rules

$$(\cap)\quad \frac{\Gamma',\ A,\ \Delta \quad ; \quad \Gamma',\ B,\ \Delta}{\Gamma',\ (A \cap B),\ \Delta}, \qquad (\neg\cap)\quad \frac{\Gamma',\ \neg A,\ \neg B,\ \Delta}{\Gamma',\ \neg(A \cap B),\ \Delta}$$

# Inference Rules of **RS**

**Implication rules**

$$(\Rightarrow)\ \frac{\Gamma^{'},\ \neg A, B,\ \Delta}{\Gamma^{'},\ (A \Rightarrow B),\ \Delta}, \qquad (\neg \Rightarrow)\ \frac{\Gamma^{'},\ A,\ \Delta \quad : \quad \Gamma^{'},\ \neg B,\ \Delta}{\Gamma^{'},\ \neg(A \Rightarrow B),\ \Delta}$$

**Negation rule**

$$(\neg\neg)\ \frac{\Gamma^{'},\ A,\ \Delta}{\Gamma^{'},\ \neg\neg A,\ \Delta}$$

where $\Gamma^{'} \in LT^{*},\ \ \Delta \in \mathcal{F}^{*},\ \ A, B \in \mathcal{F}$

# Proof System **RS**

Formally we define the system **RS** as follows

$$\textbf{RS} = (\mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}},\ \mathcal{F}^*,\ LA,\ \mathcal{R})$$

where the set of inference rules is

$$\mathcal{R} = \{(\cup),\ (\neg\cup),\ (\cap),\ (\neg\cap),\ (\Rightarrow),\ (\neg \Rightarrow),\ (\neg\neg)\}$$

and LA is the set of logical axioms

# Formal Proofs

**Definition**

By a **formal proof** of a sequence $\Gamma$ in the proof system **RS** we understand any sequence

$$\Gamma_1, \ \Gamma_2, .... \ \Gamma_n$$

of sequences of formulas (elements of $\mathcal{F}^*$, such that

$$\Gamma_1 \in LA \quad \text{and} \quad \Gamma_n = \Gamma$$

and for all $1 \leq i \leq n$

$\Gamma_i \in AL$, or $\Gamma_i$ is a **conclusion** of one of the inference rules of **RS** with all its **premisses** placed in the sequence $\Gamma_1 \Gamma_2, \ \ldots, \ \Gamma_{i-1}$

## Formal Proofs

When he proof system under consideration is fixed, we will write, as usual,

$$\vdash \Gamma$$

instead of $\vdash_{\mathbf{RS}} \Gamma$ to denote that $\Gamma$ has a **formal proof** in **RS**

As the proofs in **RS** are sequences (definition of the formal proof) of sequences of formulas (definition of **RS** ) we will not use "," to separate the steps of the proof, and write the **formal proof** as

$$\Gamma_1; \ \Gamma_2; \ \ldots. \ ; \Gamma_n$$

# Formal Proofs

We write, however, the **formal proofs** in **RS** in a form of **trees** rather then in a form of sequences

We write them in form of a **tree**, where

all leafs of the tree are **axioms**

nodes are sequences such that each sequence on the **tree** tree follows from the ones immediately preceding it by one of the **rules**

The root is a **theorem**

We picture, and write the **tree proofs** with the node on the **top**, and leafs on the very **bottom**

We adopt hence the following definition

# Proof Trees

**Definition**

By a **proof tree** in **RS** of $\Gamma$ we understand a tree

$$\mathbf{T}_{\Gamma}$$

built out of $\Gamma \in \mathcal{E}$ satisfying the following conditions:

**1.** The topmost sequence, i.e the root of $\mathbf{T}_{\Gamma}$ is the sequence $\Gamma$

**2.** all leafs are axioms

**2.** the nodes are sequences such that each sequence on the tree follows from the ones immediately preceding it by one of the **inference rules**

# Proof Trees

We picture, and write our proof trees with the **root** on the top, and the **leafs** on the very bottom,

**Additionally** we write our proof trees indicating the name of the inference rule used at each step of the proof

**Example**

Assume that a **proof** of a sequence $\Gamma$ from axioms

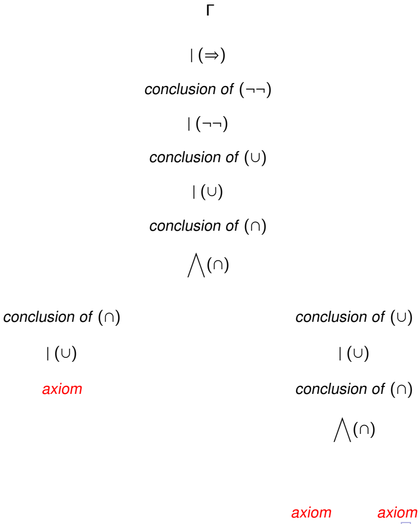
was obtained by the subsequent use of the rules

$(\cap), (\cup), (\cup), (\cap), (\cup)$, and $(\neg\neg)$, $(\Rightarrow)$

We represent it as the following tree

# Proof Trees

## The tree $\mathbf{T}_{\Gamma}$



$\Gamma$

$\mid (\Rightarrow)$

*conclusion of* $(\neg\neg)$

$\mid (\neg\neg)$

*conclusion of* $(\cup)$

$\mid (\cup)$

*conclusion of* $(\cap)$

$\bigwedge(\cap)$

*conclusion of* $(\cap)$ *conclusion of* $(\cup)$

$\mid (\cup)$ $\mid (\cup)$

*axiom* *conclusion of* $(\cap)$

$\bigwedge(\cap)$

*axiom* *axiom*

# Proof Trees

The **Proof Trees** represent a certain visualization for the proofs

Any **formal proof** in any proof system can be represented in a tree form and vice- versa

Any proof tree can be re-written in a linear form as a previously defined **formal proof**

**Example**

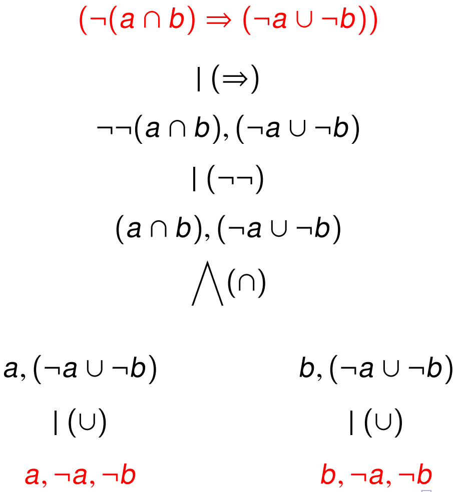
The proof tree in **RS** of the de Morgan Law

$$A = (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

is the as follows

# Proof Trees

The proof tree $\mathbf{T}_A$



$$(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\Rightarrow)$$

$$\neg\neg(a \cap b), (\neg a \cup \neg b)$$

$$\mid (\neg\neg)$$

$$(a \cap b), (\neg a \cup \neg b)$$

$$\bigwedge(\cap)$$

$$a, (\neg a \cup \neg b) \qquad b, (\neg a \cup \neg b)$$

$$\mid (\cup) \qquad \mid (\cup)$$

$$a, \neg a, \neg b \qquad b, \neg a, \neg b$$

# Formal Proof

To obtain a formal proof (written in a vertical form) of $A$ we just write down the proof tree as a sequence, starting from the **leafs** and going up (from left to right) to the **root**

$$a, \neg a, \neg b$$

$$b, \neg a, \neg b$$

$$a, (\neg a \cup \neg b)$$

$$b, (\neg a \cup \neg b$$

$$(a \cap b), (\neg a \cup \neg b)$$

$$\neg\neg(a \cap b), (\neg a \cup \neg b)$$

$$(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

# Example

**Example**

A search for the proof in **RS** of other de Morgan Law

$$A = (\neg(a \cup b) \Rightarrow (\neg a \cap \neg b))$$

consists of building a certain tree and proceeds as follows.

## Example

The tree **$\mathbf{T}_A$**

$$(\neg(a \cup b) \Rightarrow (\neg a \cap \neg b))$$

$$\mid (\Rightarrow)$$

$$\neg\neg(a \cup b), (\neg a \cap \neg b)$$

$$\mid (\neg\neg)$$

$$(a \cup b), (\neg a \cap \neg b)$$

$$\mid (\cup)$$

$$a, b, (\neg a \cap \neg b)$$

$$\bigwedge (\cap)$$

$$a, b, \neg a \qquad\qquad a, b, \neg b$$

# Example

We construct its formal proof, as before, written in a vertical manner as follows

$$a, b, \neg b$$

$$a, b, \neg a$$

$$a, b, (\neg a \cap \neg b)$$

$$(a \cup b), (\neg a \cap \neg b)$$

$$\neg\neg(a \cup b), (\neg a \cap \neg b)$$

$$(\neg(a \cup b) \Rightarrow (\neg a \cap \neg b))$$

# Decomposition Trees

The **goal** in inventing proof systems like **RS** is to facilitates **automatic** proof search

The **method** of suchproof search is to generate what is called the **decomposition trees**

A **decomposition tree** $\mathbf{T}_A$ for the formula

$$A = (((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c))$$

is built as follows

# Decomposition Trees

$$\mathbf{T}_A$$

$$(((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c))$$

$$\mid (\cup)$$

$$((a \Rightarrow b) \cap \neg c), (a \Rightarrow c)$$

$$\bigwedge (\cap)$$

$$(a \Rightarrow b), (a \Rightarrow c)$$

$$\mid (\Rightarrow)$$

$$\neg a, b, (a \Rightarrow c)$$

$$\mid (\Rightarrow)$$

$$\neg a, b, \neg a, c$$

$$\neg c, (a \Rightarrow c)$$

$$\mid (\Rightarrow)$$

$$\neg c, \neg a, c$$

# **RS** Decomposition Rules
# and
# Decomposition Trees

# Decomposition Trees

The process of searching for a proof of a formula $A \in \mathcal{F}$ in **RS** consists of building a certain tree $\mathbf{T}_A$, called a **decomposition tree**

Building a **decomposition tree** what really is a proof search **tree** consists in the **first step** of transforming the **RS rules** into corresponding decomposition **rules**

# Decomposition Rules

## **RS** Decomposition Rules

### Disjunction

$$(\cup)\ \frac{\Gamma',\ (A \cup B),\ \Delta}{\Gamma',\ A, B,\ \Delta}, \qquad (\neg\cup)\ \frac{\Gamma',\ \neg(A \cup B),\ \Delta}{\Gamma',\ \neg A,\ \Delta \quad ; \quad \Gamma',\ \neg B,\ \Delta}$$

### Conjunction

$$(\cap)\ \frac{\Gamma',\ (A \cap B),\ \Delta}{\Gamma', A, \Delta \quad ; \quad \Gamma',\ B, \Delta}, \qquad (\neg\cap)\ \frac{\Gamma',\ \neg(A \cap B),\ \Delta}{\Gamma',\ \neg A, \neg B,\ \Delta}$$

# Decomposition Rules

**Implication**

$$(\Rightarrow)\ \frac{\Gamma^{'},\ (A\Rightarrow B),\ \Delta}{\Gamma',\ \neg A, B,\ \Delta}, \qquad (\neg\Rightarrow)\ \frac{\Gamma^{'},\ \neg(A\Rightarrow B),\ \Delta}{\Gamma', A, \Delta\quad ;\quad \Gamma',\ \neg B,\ \Delta}$$

**Negation**

$$(\neg\neg)\ \frac{\Gamma^{'},\ \neg\neg A,\ \Delta}{\Gamma',\ A,\ \Delta}$$

where $\Gamma^{'} \in \mathcal{F}'^{*},\ \ \Delta \in \mathcal{F}^{*},\ \ A, B \in \mathcal{F}$

# Tree Rules

We write the **Decomposition Rules** in a visual tree form as follows

**Tree Rules**

$(\cup)$ **rule**

$$\Gamma^{'},\ (A \cup B),\ \Delta$$

$$\mid (\cup)$$

$$\Gamma^{'},\ A, B,\ \Delta$$

# Tree Rules

**(¬∪) rule**

$$\Gamma', \neg(A \cup B), \Delta$$

$$\bigwedge(\neg\cup)$$

$$\Gamma', \neg A, \Delta \qquad\qquad \Gamma', \neg B, \Delta$$

**(∩) rule**

$$\Gamma', (A \cap B), \Delta$$

$$\bigwedge(\cap)$$

$$\Gamma', A, \Delta \qquad\qquad \Gamma', B, \Delta$$

# Tree Rules

$(\neg\cup)$ **rule**

$$\Gamma^{'},\ \neg(A \cap B),\ \Delta$$

$$\mid (\neg\cap)$$

$$\Gamma^{'},\ \neg A, \neg B,\ \Delta$$

$(\Rightarrow)$ **rule**

$$\Gamma^{'},\ (A \Rightarrow B),\ \Delta$$

$$\mid (\Rightarrow)$$

$$\Gamma^{'},\ \neg A, B,\ \Delta$$

# Tree Rules

$(\neg \Rightarrow)$ **rule**

$$\Gamma^{'}, \neg(A \Rightarrow B), \Delta$$

$$\bigwedge(\neg \Rightarrow)$$

$$\Gamma^{'}, A, \Delta \qquad\qquad \Gamma^{'}, \neg B, \Delta$$

$(\neg\neg)$ **rule**

$$\Gamma^{'}, \neg\neg A, \Delta$$

$$\mid (\neg\neg)$$

$$\Gamma^{'}, A, \Delta$$

## Definitions and Observations

**Observe** that we use the same names for the **inference** and **decomposition** rules

We do so because once the we have built the **decomposition tree** with all leaves being **axioms**, it constitutes a **proof** of $A$ in **RS** with branches labeled by the proper **inference rules**

Now we still need to introduce few standard and useful definitions and observations.

## Definitions and Observations

**Definition**

A sequence $\Gamma'$ built only out of literals, i.e. $\Gamma \in \mathcal{F}'^{*}$ is called an **indecomposable sequence**

**Definition**

A formula $A$ that is not a literal, i.e. $A \in \mathcal{F} - LT$ is called a **decomposable formula**

**Definition**

A sequence $\Gamma$ that contains a decomposable formula is called a

**decomposable sequence**

## Definitions and Observations

**Observation 1**

For any **decomposable** sequence, i.e. for any $\Gamma \notin LT^*$ there is **exactly one** decomposition rule that can be applied to it

This rule is **determined** by the first **decomposable formula** in $\Gamma$ and by the main connective of that formula

# Definitions and Observations

**Observation 2**

If the main connective of the **first** decomposable formula is $\cup, \cap, \Rightarrow$, then the **decomposition rule** determined by it is $(\cup), (\cap), (\Rightarrow)$, respectively

**Observation 3**

If the main connective of the **first** decomposable formula $A$ is negation $\neg$, then the **decomposition rule** is determined by the **second connective** of the formula $A$

The corresponding **decomposition rules** are $(\neg\cup), (\neg\cap), (\neg\neg), (\neg \Rightarrow)$

# Decomposition Lemma

Because of the importance of the **Observation 1** we re-write it in a form of the following

**Decomposition Lemma**

For any sequence $\Gamma \in \mathcal{F}^*$,

$\Gamma \in LT^*$ or $\Gamma$ is in the domain of **exactly one** of **RS** Decomposition Rules

This rule is **determined** by the first decomposable formula in $\Gamma$ and by the main connective of that formula

# Decomposition Tree Definition

**Definition:** **Decomposition Tree** $\mathbf{T}_A$

For each $A \in \mathcal{F}$, a **decomposition tree** $\mathbf{T}_A$ is a tree build as follows

**Step 1.**

The formula $A$ is the **root** of $\mathbf{T}_A$

For any other **node** $\Gamma$ of the tree we follow the steps below

**Step 2.**

If $\Gamma$ is **indecomposable** then $\Gamma$ becomes a **leaf** of the tree

# Decomposition Tree Definition

**Step 3.**

If $\Gamma$ is **decomposable**, then we **traverse** $\Gamma$ from **left** to **right** and identify the **first** **decomposable formula** $B$

By the **Decomposition Lemma**, there is exactly one decomposition rule determined by the main connective of $B$

**We put** its premiss as a **node below**, or its left and right premisses as the left and right **nodes below**, respectively

**Step 4.**

We repeat **Step 2** and **Step 3** until we obtain only leaves

# Decomposition Theorem

We now prove the following **Decomposition Tree Theorem**. This Theorem provides a crucial step in the proof of the Completeness Theorem for RS

**Decomposition Tree Theorem**

For any sequence $\Gamma \in \mathcal{F}^*$ the following conditions hold

**1.** $\mathbf{T}_\Gamma$ is finite and unique

**2.** $\mathbf{T}_\Gamma$ is a proof of $\Gamma$ in **RS** if and only if all its leafs are axioms

**3.** $\nvdash_{\mathbf{RS}} \Gamma$ if and only if $\mathbf{T}_\Gamma$ has a non- axiom leaf

# Theorem

**Proof**

The tree $\mathbf{T}_\Gamma$ is unique by the **Decomposition Lemma**

It is finite because there is a finite

number of logical connectives in $\Gamma$ and all decomposition rules diminish the

number of connectives

If the tree $\mathbf{T}_\Gamma$ has a **non- axiom** leaf it is not a proof by

definition

By **1.** it also means that the proof **does not exist**

# Example

**Example**

Let's construct, as an example a decomposition tree $\mathbf{T}_A$ of the following formula $A$

$$(((a \cup b) \Rightarrow \neg a) \cup (\neg a \Rightarrow \neg c))$$

The formula A forms a one element **decomposable sequence**

The first decomposition rule used is determined by its main connective

We put a **box** around it, to make it more visible

$$(((a \cup b) \Rightarrow \neg a) \boxed{\cup} (\neg a \Rightarrow \neg c))$$

# Example

The first and only decomposition rule to be applied is $(\cup)$

The first segment of the decomposition tree $\mathbf{T}_A$ is

$$(((a \cup b) \Rightarrow \neg a) \boxed{\cup} (\neg a \Rightarrow \neg c))$$

$$\mid (\cup)$$

$$((a \cup b) \Rightarrow \neg a), \; (\neg a \Rightarrow \neg c)$$

## Example

Now we decompose the sequence

$$((a \cup b) \Rightarrow \neg a),\ (\neg a \Rightarrow \neg c)$$

It is a **decomposable** sequence with the first, decomposable formula

$$((a \cup b) \Rightarrow \neg a)$$

The next step of the construction of our decomposition tree is determined by its main connective $\Rightarrow$ and we put the box around it

$$((a \cup b)\boxed{\Rightarrow}\neg a),\ (\neg a \Rightarrow \neg c)$$

## Example

The decomposition tree becomes now

$$(((a \cup b) \Rightarrow \neg a) \boxed{\cup} (\neg a \Rightarrow \neg c))$$

$$\mid (\cup)$$

$$((a \cup b) \boxed{\Rightarrow} \neg a), (\neg a \Rightarrow \neg c)$$

$$\mid (\Rightarrow)$$

$$\neg(a \cup b), \neg a, (\neg a \Rightarrow \neg c)$$

# Example

The next sequence to decompose is

$$\neg(a \cup b), \neg a, (\neg a \Rightarrow \neg c)$$

with the first decomposable formula

$$\neg(a \cup b)$$

Its main connective is $\neg$, so to find the appropriate rule we have to examine next connective, which is $\cup$

The **decomposition rule** determine by this stage of decomposition is $(\neg\cup)$

# Example

Next stage of the construction of the decomposition tree **$\mathbf{T}_A$** is

$$(((a \cup b) \Rightarrow \neg a) \boxed{\cup} (\neg a \Rightarrow \neg c))$$

$$\mid (\cup)$$

$$((a \cup b) \boxed{\Rightarrow} \neg a), (\neg a \Rightarrow \neg c)$$

$$\mid (\Rightarrow)$$

$$\boxed{\neg} (a \boxed{\cup} b), \neg a, (\neg a \Rightarrow \neg c)$$

$$\bigwedge (\neg \cup)$$

$$\neg a, \neg a, (\neg a \Rightarrow \neg c) \qquad\qquad \neg b, \neg a, (\neg a \Rightarrow \neg c)$$

# Example

Finally, the complete $\mathbf{T}_A$ is

$$(((a \cup b) \Rightarrow \neg a) \boxed{\cup} (\neg a \Rightarrow \neg c))$$

$$\mid (\cup)$$

$$((a \cup b) \boxed{\Rightarrow} \neg a), (\neg a \Rightarrow \neg c)$$

$$\mid (\Rightarrow)$$

$$\boxed{\neg} (a \boxed{\cup} b), \neg a, (\neg a \Rightarrow \neg c)$$

$$\bigwedge (\neg \cup)$$

$$\neg a, \neg a, (\neg a \boxed{\Rightarrow} \neg c)$$

$$\mid (\Rightarrow)$$

$$\neg a, \neg a, \boxed{\neg\neg} a, \neg c$$

$$\mid (\neg\neg)$$

$$\neg a, \neg a, a, \neg c$$

$$\neg b, \neg a, (\neg a \boxed{\Rightarrow} \neg c)$$

$$\mid (\Rightarrow)$$

$$\neg b, \neg a, \boxed{\neg\neg} a, \neg c$$

$$\mid (\neg\neg)$$

$$\neg b, \neg a, a, \neg c$$

# Example

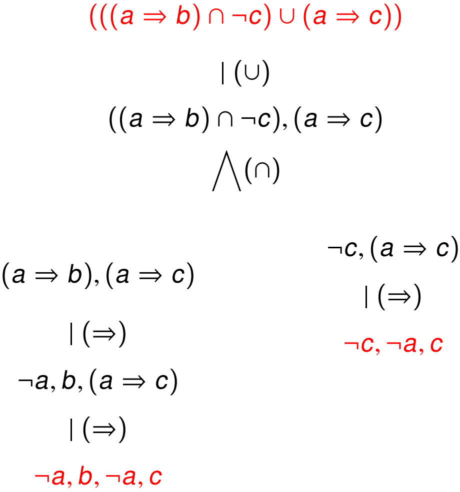
All leaves of $\mathbf{T}_A$ are axioms

The tree $\mathbf{T}_A$ is a **proof** of $A$ in **RS**, i.e.

$$\vdash_{\mathbf{RS}} (((a \cup b) \Rightarrow \neg a) \cup (\neg a \Rightarrow \neg c))$$

# Example

**Example** Given a formula $A$ and its decomposition tree $\mathbf{T}_A$



$$(((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c))$$

$$\mid (\cup)$$

$$((a \Rightarrow b) \cap \neg c), (a \Rightarrow c)$$

$$\bigwedge (\cap)$$

$$(a \Rightarrow b), (a \Rightarrow c)$$

$$\mid (\Rightarrow)$$

$$\neg a, b, (a \Rightarrow c)$$

$$\mid (\Rightarrow)$$

$$\neg a, b, \neg a, c$$

$$\neg c, (a \Rightarrow c)$$

$$\mid (\Rightarrow)$$

$$\neg c, \neg a, c$$

# Example

There is a leaf $\neg a, b, \neg a, c$ of the tree $\mathbf{T}_A$ that is **not an axiom**

By the **Decomposition Tree Theorem**

$$\nvdash_{\mathbf{RS}} (((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c))$$

It means that the **proof** in **RS** of the formula $(((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c))$ **does not exists**

# Completeness Theorem

Our main goal is to prove the **Completeness Theorem** for **RS**

We **prove** first the following **Completeness Theorem** for formulas $A \in \mathcal{F}$

**Completeness Theorem 1** For any formula $A \in \mathcal{F}$

$$\vdash_{\mathbf{RS}} A \quad \text{if and only if} \quad \models A$$

and then we generalize it to the following

**Completeness Theorem 2** For any $\Gamma \in \mathcal{F}^*$,

$$\vdash_{\mathbf{RS}} \Gamma \quad \text{if and only if} \quad \models \Gamma$$

Do do so we need to introduce a new notion of a Strong Soundness and prove that the **RS** is strongly sound

# Part 2: Strong Soundness
and
Constructive Completeness

# Strong Soundness

**Definition**

Given a proof system

$$S = (\mathcal{L}, \ \mathcal{E}, \ LA, \ \mathcal{R})$$

**Definition**

A rule $r \in \mathcal{R}$ such that the **conjunction** of all its premisses is **logically equivalent** to its conclusion is called **strongly sound**

**Definition**

A proof system $S$ is called **strongly sound** if and only if $S$ is sound and **all** its rules $r \in \mathcal{R}$ are **strongly sound**

## Strong Soundness of RS

**Theorem**

The proof system **RS** is strongly sound

**Proof**

We prove as an example the **strong soundness** of two of inference rules: $(\cup)$ and $(\neg\cup)$

Proof for all other rules follows the same patterns and is left as an exercise

By definition of strong soundness we have to show that

If $P_1$, $P_2$ are premisses of a given rule and $C$ is its conclusion, then for all v,

$$v^*(P_1) = v^*(C)$$

in case of one premiss rule and

$$v^*(P_1) \cap v^*(P_2) = v^*(C)$$

in case of the two premisses rule.

# Strong Soundness of RS

Consider the rule $(\cup)$

$$(\cup) \quad \frac{\Gamma',\ A, B,\ \Delta}{\Gamma',\ (A \cup B),\ \Delta}$$

We evaluate:

$$v^*(\Gamma^{'}, A, B, \Delta) = v^*(\delta_{\{\Gamma', A, B, \Delta\}}) = v^*(\Gamma^{'}) \cup v^*(A) \cup v^*(B) \cup v^*(\Delta)$$

$$= v^*(\Gamma^{'}) \cup v^*(A \cup B) \cup v^*(\Delta) = v^*(\delta_{\{\Gamma', (A \cup B), \Delta\}})$$

$$= v^*(\Gamma^{'}, (A \cup B), \Delta)$$

# Strong Soundness of RS

Consider the rule $(\neg\cup)$

$$(\neg\cup)\ \frac{\Gamma',\ \neg A,\ \Delta \quad : \quad \Gamma',\ \neg B,\ \Delta}{\Gamma',\ \neg(A\cup B),\ \Delta}$$

We evaluate:

$$v^*(P_1)\cap v^*(P_2) = v^*(\Gamma',\neg A,\Delta)\cap v^*(\Gamma',\neg B,\Delta)$$

$$= (v^*(\Gamma')\cup v^*(\neg A)\cup v^*(\Delta))\cap(v^*(\Gamma')\cup v^*(\neg B)\cup v^*(\Delta))$$

$$= (v^*(\Gamma',\Delta)\cup v^*(\neg A))\cap(v^*(\Gamma',\Delta)\cup v^*(\neg B))$$

$$=^{distrib}\ (v^*(\Gamma',\Delta)\cup(v^*(\neg A)\cap v^*(\neg B))$$

$$= v^*(\Gamma')\cup v^*(\Delta)\cup v^*(\neg A\cap\neg B) =^{deMorgan}\ v^*(\delta_{\{\Gamma',\neg(A\cup B),\Delta\}}$$

$$= v^*(\Gamma',\ \neg(A\cup B),\ \Delta) = v^*(C)$$

# Soundness Theorem

**Observe** that the strong soundness notion implies soundness (not only by name!). Obviously the LA of **RS** are **tautologies** , hence we have also proved the following

**Soundness Theorem** for **RS**

For any $\Gamma \in \mathcal{F}^*$,

$$\text{If} \quad \vdash_{\mathbf{RS}} \Gamma, \quad \text{then} \quad \models \Gamma$$

In particular, for any $A \in \mathcal{F}$,

$$\text{If} \quad \vdash_{\mathbf{RS}} A, \quad \text{then} \quad \models A$$

# Strong Soundness

We proved that all the rules of inference of **RS** of are **strongly sound**, i.e. $C \equiv P$ and $C \equiv P_1 \cap P_2$

**Strong soundness** of the rules hence means that if **at least** one of premisses of a rule is **false**, so is its conclusion

Given a formula A, such that its $\mathbf{T}_A$ has a branch ending with a non-axiom leaf

By **strong soundness**, any v that make this non-axiom leaf **false** also **falsifies** all sequences on that branch, and hence **falsifies the** the formula A

This means that any v that **falsifies** a non-axiom leaf is a **counter-model** for A

# Counter Model Theorem

We have proved the following

**Counter Model Theorem**

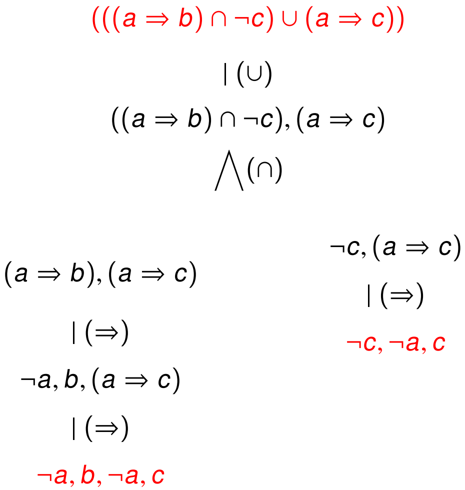
Let $A \in \mathcal{F}$ be such that its decomposition tree $\mathbf{T}_A$ contains a **non- axiom** leaf $L_A$

Any truth assignment v that **falsifies** $L_A$ is a **counter model** for A

Any truth assignment that **falsifies** a non- axiom **leaf** is called a **counter-model** for $A$ **determined** by the decomposition tree $\mathbf{T}_A$

# Counter Model Example

Consider a tree $\mathbf{T}_A$



$$(((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c))$$

$$\mid (\cup)$$

$$((a \Rightarrow b) \cap \neg c), (a \Rightarrow c)$$

$$\bigwedge (\cap)$$

$$(a \Rightarrow b), (a \Rightarrow c)$$

$$\mid (\Rightarrow)$$

$$\neg a, b, (a \Rightarrow c)$$

$$\mid (\Rightarrow)$$

$$\neg a, b, \neg a, c$$

$$\neg c, (a \Rightarrow c)$$

$$\mid (\Rightarrow)$$

$$\neg c, \neg a, c$$

## Counter Model Example

The tree $\mathbf{T}_A$ has a **non-axiom leaf**

$$L_A : \neg a,\ b, \neg a,\ c$$

We want to define a truth assignment $v : \ VAR \longrightarrow \{T, F\}$ **falsifies** this leaf $L_A$

Observe that v must be such that

$v^*(\neg a, b, \neg a, c) = v^*(\neg a) \cup v^*(b) \cup v^*(\neg a) \cup v^*(c) =$

$\neg v(a) \cup v(b) \cup \neg v(a) \cup v(c) = F$

It means that all components of the **disjunction** must be put to F

# Counter Model Example

We hence get that v must be such that

$$v(a) = T, \; v(b) = F, \;\; v(c) = F$$

By the **Counter Model Theorem**, the v **determined** by the non-axiom leaf also **falsifies** the formula A

IT proves that v is a **counter model** for A and

$$\not\models (((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c))$$

# Counter Model

The **Counter Model Theorem** says that **F** determined by the non-axiom leaf "climbs" the tree $\mathbf{T}_A$

$$(((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c)) = \mathbf{F}$$

$$\mid (\cup)$$

$$((a \Rightarrow b) \cap \neg c), (a \Rightarrow c) = \mathbf{F}$$

$$\bigwedge (\cap)$$

$$(a \Rightarrow b), (a \Rightarrow c) = \mathbf{F}$$

$$\mid (\Rightarrow)$$

$$\neg a, b, (a \Rightarrow c) = \mathbf{F}$$

$$\mid (\Rightarrow)$$

$$\neg a, b, \neg a, c = \mathbf{F}$$

$$\neg c, (a \Rightarrow c)$$

$$\mid (\Rightarrow)$$

$$\neg c, \neg a, c$$

*axiom*

# Counter Model

**Observe** that the same counter model **construction** applies to any other non-axiom **leaf**, if exists

The other non-axiom leaf defines another **F** that also ”climbs the tree” picture, and hence defines another **counter- model** for $A$

By **Decomposition Tree Theorem** all possible **restricted** counter-models for $A$ are those **determined** by all non-axioms **leaves** of the $\mathbf{T}_A$

In our case the formula $\mathbf{T}_A$ has only one non-axiom leaf, and hence only one restricted **counter model**

# **RS** Completeness Theorem

**RS Completeness Theorem**

For any $A \in \mathcal{F}$,

$$\text{If} \quad \models A, \quad \text{then} \quad \vdash_{\mathbf{RS}} A$$

We prove instead the opposite implication

**RS Completeness Theorem**

$$\text{If} \quad \nvdash_{\mathbf{RS}} A \quad \text{then} \quad \not\models A$$

# Proof of Completeness Theorem

**Proof** of **Completeness Theorem**

Assume that $A$ is any formula is such that

$$\nvdash_{\mathbf{RS}} A$$

By the **Decomposition Tree Theorem** the $\mathbf{T}_A$ contains a non-axiom leaf

The non-axiom leaf $L_A$ **defines** a truth assignment $v$ which **falsifies** it as follows:

$$v(a) = \begin{cases} F & \text{if } a \text{ appears in } L_A \\ T & \text{if } \neg a \text{ appears in } L_A \\ \text{any value} & \text{if } a \text{ does not appear in } L_A \end{cases}$$

Hence by **Counter Model Theorem** we have that $v$ also **falsifies** $A$ , i.e.

$$\not\models A$$

# PART3:
# Proof Systems **RS1** and **RS2**

# RS1 Proof System

Poof System **RS1**

**Language** of **RS1** is the same as the language of **RS** i.e.

$$\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$$

**Expressions**

$$\mathcal{E} = \mathcal{F}^*$$

is the set of expressions of **RS1**

**Notation**

Elements of $\mathcal{E}$ are finite sequences of formulas and we denote them by

$$\Gamma, \Delta, \Sigma \ldots$$

with indices if necessary.

# Rules of inference of RS1

**Rules of inference**

**RS1** contains **seven inference rules**, denoted by the same symbols as the rules of **RS**

$$(\cup), \quad (\neg\cup), \quad (\cap), \quad (\neg\cap), \quad (\Rightarrow), \quad (\neg \Rightarrow), \quad (\neg\neg)$$

The inference rules of **RS1** are quite similar to the rules of **RS**

Observe them **carefully** to see where lies the difference

**Reminder**

Any propositional variable, or a negation of a propositional variable is called a **literal**

The set

$$LT = VAR \cup \{\neg a : \quad a \in VAR\}$$

is called a set of all propositional **literals**

# Literals Notation

We denote, as before, by

$$\Gamma^{'}, \quad \Delta^{'}, \quad \Sigma^{'} \ldots$$

finite sequences (empty included) formed out of **literals** i.e

$$\Gamma^{'}, \Delta^{'}, \Sigma^{'} \in LT^{*}$$

We will denote by

$$\Gamma, \quad \Delta, \quad \Sigma \ldots$$

the elements of $\mathcal{F}^{*}$

# Logical Axioms

**Logical Axioms**

We adopt all logical axioms of **RS** as the axioms of **RS1**, i.e.

$$\Gamma'_1,\ a,\ \Gamma'_2,\ \neg a,\ \Gamma'_3$$

$$\Gamma'_1,\ \neg a,\ \Gamma'_2,\ a,\ \Gamma'_3$$

where $a \in VAR$ is any **propositional variable**

# Inference Rules of **RS1**

**Disjunction rules**

$$(\cup)\ \frac{\Gamma,\ A, B,\ \Delta'}{\Gamma,\ (A \cup B),\ \Delta'} \qquad (\neg\cup)\ \frac{\Gamma,\ \neg A,\ \Delta' \quad ; \quad \Gamma,\ \neg B,\ \Delta'}{\Gamma,\ \neg(A \cup B),\ \Delta'}$$

**Conjunction rules**

$$(\cap)\ \frac{\Gamma,\ A,\ \Delta' \quad ; \quad \Gamma,\ B,\ \Delta'}{\Gamma,\ (A \cap B),\ \Delta'} \qquad (\neg\cap)\ \frac{\Gamma,\ \neg A,\ \neg B,\ \Delta'}{\Gamma,\ \neg(A \cap B),\ \Delta'}$$

# Inference Rules of **RS1**

**Implication rules**

$$(\Rightarrow)\ \frac{\Gamma^{'},\ \neg A, B,\ \Delta^{'}}{\Gamma^{'},\ (A \Rightarrow B),\ \Delta^{'}} \qquad (\neg \Rightarrow)\ \frac{\Gamma^{'},\ A,\ \Delta^{'} \quad : \quad \Gamma^{'},\ \neg B,\ \Delta^{'}}{\Gamma^{'},\ \neg(A \Rightarrow B),\ \Delta^{'}}$$

**Negation rule**

$$(\neg\neg)\ \frac{\Gamma^{'},\ A,\ \Delta^{'}}{\Gamma^{'},\ \neg\neg A,\ \Delta^{'}}$$

where $\Gamma^{'} \in LT^{*},\ \ \Delta \in \mathcal{F}^{*},\ \ A, B \in \mathcal{F}$

# Proof System **RS1**

Formally we define the system **RS1** as follows

$$\textbf{RS1} = (\mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}},\ \ \mathcal{E},\ \ LA,\ \ \mathcal{R})$$

where

$$\mathcal{R} = \{(\cup),\ (\neg\cup),\ (\cap),\ (\neg\cap),\ (\Rightarrow),\ (\neg\Rightarrow),\ (\neg\neg)\}$$

for the inference rules is defined above and LA is the set of all logical axioms is the same as for **RS**

# System **RS1**

**Exercises**

**E1.** Construct a proof in **RS1** of a formula

$$A = (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

**E2.** Prove that **RS1** is **strongly sound**

**E3.** Define in your own words, for any formula $A$, the decomposition tree $\mathbf{T}_A$ in **RS1**

**E4.** Prove **Completeness Theorem** for **RS1**

## Exercises Solutions

**E1.** The decomposition tree $\mathbf{T}_A$ is a **proof** of $A$ in **RS1** as all leaves are **axioms**

$$\mathbf{T}_A$$

$$(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\Rightarrow)$$

$$\neg\neg(a \cap b), (\neg a \cup \neg b)$$

$$\mid (\cup)$$

$$\neg\neg(a \cap b), \neg a, \neg b$$

$$\mid (\neg\neg)$$

$$(a \cap b), \neg a, \neg b$$

$$\bigwedge(\cap)$$

$$a, \neg a, \neg b \qquad\qquad b, \neg a, \neg b$$

# Exercises Solutions

**E2.** Prove that **RS1** is **strongly sound**

Observe that the system **RS1** is obtained from **RS** by changing the sequence $\Gamma^{'}$ into $\Gamma$ and the sequence $\Delta$ into $\Delta^{'}$ in **all** of the rules of inference of **RS**

These changes do not influence the essence of proof of **strong soundness** of the rules of **RS**

One has just to replace the sequence $\Gamma^{'}$ by $\Gamma$ and $\Delta$ by $\Delta^{'}$ in the the proof of **strong soundness** of each rule of **RS** to obtain the corresponding proof of **strong soundness** of corresponding rule of **RS1**

# Strong Soundness of **RS1**

We do it, for example for the rule $(\cup)$ as follows

$$(\cup)\quad \frac{\Gamma,\ A, B,\ \Delta^{'}}{\Gamma,\ (A\cup B),\ \Delta^{'}}$$

We evaluate:

$$v^*(\Gamma, A, B, \Delta^{'}) = v^*(\delta_{\{\Gamma,A,B,\Delta'\}}) = v^*(\Gamma)\cup v^*(A)\cup v^*(B)\cup v^*(\Delta^{'})$$

$$= v^*(\Gamma)\cup v^*(A\cup B)\cup v^*(\Delta^{'}) = v^*(\delta_{\{\Gamma,(A\cup B),\Delta'\}})$$

$$= v^*(\Gamma, (A\cup B), \Delta^{'})$$

# Decomposition Trees in **RS1**

**E3.** Define in your own words, for any formula $A$, the decomposition tree $\mathbf{T}_A$ in **RS1**

The **definition** of the decomposition tree $\mathbf{T}_A$ is in its essence similar to the one for **RS** except for the changes which reflect the **differences** in the corresponding rules of inference

# Decomposition Trees in **RS1**

**Definition**

To construct the decomposition tree $\mathbf{T}_A$ we follow the steps below

**Step 1**

Decompose formula $A$ using a rule defined by its main connective

**Step 2**

Traverse resulting sequence $\Gamma$ on the new node of the tree from **right** to **left** and **find** the first decomposable formula

**Step 3**

Repeat **Step 1** and **Step 2** until there is **no more** decomposable formulas

**End** of the decomposition tree construction

# Completeness Theorem for **RS1**

**E4.** Prove the following **Completeness Theorem**

For any $A \in \mathcal{F}$,

$$\text{If} \quad \models A, \quad \text{then} \quad \vdash_{\mathbf{RS1}} A$$

We prove instead the opposite implication

**Completeness Theorem**

$$\text{If} \quad \nvdash_{\mathbf{RS1}} A \quad \text{then} \quad \not\models A$$

# Completeness Theorem for **RS1**

Observe that directly from the definition of the decomposition tree $\mathbf{T}_A$ we have that the following holds

**Fact 1:** The decomposition tree $\mathbf{T}_A$ is a **proof** if and only if all leaves are **axioms**

**Fact 2:** The proof does not exist otherwise, i.e.
$\nvdash_{\mathbf{RS1}} A$ if and only if
there is a **non- axiom leaf** on $\mathbf{T}_A$

**Fact 2** holds because the tree $\mathbf{T}_A$ is unique

# Proof of Completeness Theorem for **RS1**

Observe that we need **Facts 1, 2** in order to prove the **Completeness Theorem** by construction of a counter-model generated by a the a non- axiom leaf

**Proof**

Assume that $A$ is any formula such that

$$\nvdash_{\mathbf{RS1}} A$$

By **Fact 2** the decomposition tree $\mathbf{T}_A$ contains a non-axiom leaf $L_A$

We use the non-axiom leaf $L_A$ and **define** a truth assignment $v$ which falsifies $A$ as follows:

$$v(a) = \begin{cases} F & \text{if } a \text{ appears in } L_A \\ T & \text{if } \neg a \text{ appears in } L_A \\ \text{any value} & \text{if } a \text{ does not appear in } L_A \end{cases}$$

This proves that

$$\not\models A$$

# System **RS2** Definition

**RS2 Definition**

System **RS2** is a proof system obtained from **RS** by changing the sequences $\Gamma'$ into $\Gamma$ in **all of the rules** of inference of **RS**

The **logical axioms** LA remind the same

Observe that now the decomposition tree may not be unique

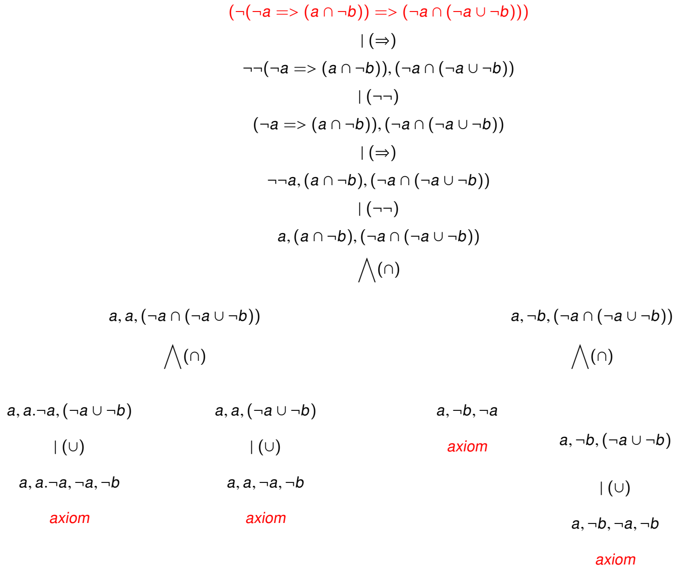
**Exercise 1**

Construct **two** decomposition trees in **RS2** of the formula

$$(\neg(\neg a \Rightarrow (a \cap \neg b)) \Rightarrow (\neg a \cap (\neg a \cup \neg b)))$$

# **RS2** Exercises

**T1**$_A$



$(\neg(\neg a => (a \cap \neg b)) => (\neg a \cap (\neg a \cup \neg b)))$

$\mid (\Rightarrow)$

$\neg\neg(\neg a => (a \cap \neg b)), (\neg a \cap (\neg a \cup \neg b))$

$\mid (\neg\neg)$

$(\neg a => (a \cap \neg b)), (\neg a \cap (\neg a \cup \neg b))$

$\mid (\Rightarrow)$

$\neg\neg a, (a \cap \neg b), (\neg a \cap (\neg a \cup \neg b))$

$\mid (\neg\neg)$

$a, (a \cap \neg b), (\neg a \cap (\neg a \cup \neg b))$

$\bigwedge (\cap)$

$a, a, (\neg a \cap (\neg a \cup \neg b))$

$\bigwedge (\cap)$

$a, a.\neg a, (\neg a \cup \neg b)$

$\mid (\cup)$

$a, a.\neg a, \neg a, \neg b$

*axiom*

$a, a, (\neg a \cup \neg b)$

$\mid (\cup)$

$a, a, \neg a, \neg b$

*axiom*

$a, \neg b, (\neg a \cap (\neg a \cup \neg b))$

$\bigwedge (\cap)$

$a, \neg b, \neg a$

*axiom*

$a, \neg b, (\neg a \cup \neg b)$

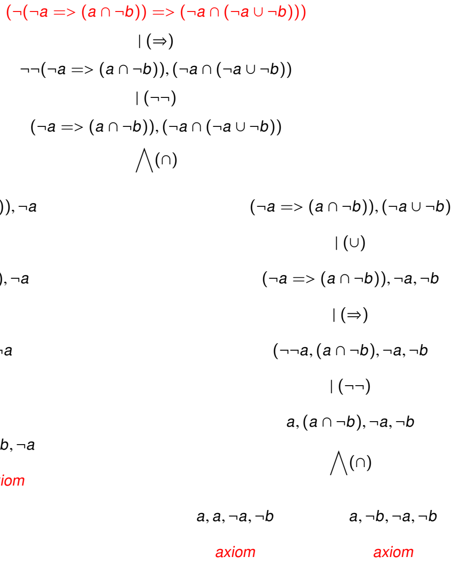
$\mid (\cup)$

$a, \neg b, \neg a, \neg b$

*axiom*

# **RS2** Exercises

**T2**$_A$



$(\neg(\neg a => (a \cap \neg b)) => (\neg a \cap (\neg a \cup \neg b)))$

$\mid (\Rightarrow)$

$\neg\neg(\neg a => (a \cap \neg b)), (\neg a \cap (\neg a \cup \neg b))$

$\mid (\neg\neg)$

$(\neg a => (a \cap \neg b)), (\neg a \cap (\neg a \cup \neg b))$

$\bigwedge (\cap)$

$(\neg a => (a \cap \neg b)), \neg a$

$\mid (\Rightarrow)$

$(\neg\neg a, (a \cap \neg b)), \neg a$

$\mid (\neg\neg)$

$a, (a \cap \neg b), \neg a$

$\bigwedge (\cap)$

$a, a, \neg a$

*axiom*

$a, \neg b, \neg a$

*axiom*

$(\neg a => (a \cap \neg b)), (\neg a \cup \neg b)$

$\mid (\cup)$

$(\neg a => (a \cap \neg b)), \neg a, \neg b$

$\mid (\Rightarrow)$

$(\neg\neg a, (a \cap \neg b), \neg a, \neg b$

$\mid (\neg\neg)$

$a, (a \cap \neg b), \neg a, \neg b$

$\bigwedge (\cap)$

$a, a, \neg a, \neg b$

*axiom*

$a, \neg b, \neg a, \neg b$

*axiom*

# System **RS2**

**Exercise 2**

Explain why the system **RS2** is **strongly sound**. You can use the soundness of the system RS

**Solution**

The only difference between **RS** and **RS2** is that in **RS2** each inference rule has at the beginning a sequence of any formulas, not only of literals, as in **RS**

So there are many ways to **apply rules** as the decomposition rules while constructing the **decomposition tree**

But it does not affect **strong soundness**, since for all rules of **RS2** premisses and conclusions are still logically equivalent as they were in **RS**

## RS2 Exercises

Consider, for example, **RS2** rule

$$(\cup)\quad \frac{\Gamma, A, B, \Delta}{\Gamma, (A \cup B), \Delta}$$

We evaluate

$$v^*(\Gamma, A, B, \Delta) = v^*(\Gamma) \cup v^*(A) \cup v^*(B) \cup v^*(\Delta) =$$

$$v^*(\Gamma) \cup v^*(A \cup B) \cup v^*(\Delta) = v^*(\Gamma, (A \cup B), \Delta)$$

Similarly, as in RS, we show all other rules of RS2 to be **strongly sound**, thus **RS2** is also **strongly sound**

# **RS2** Exercises

**Exercise 3**

Define shortly, in your own words, for any formula $A$, its **decomposition tree** $\mathbf{T}_A$ in **RS2**

**Justify** why your definition is correct

Show that in **RS2** the decomposition tree for some formula A may not be unique

# RS2 Exercises

**Solution**

Given a formula $A$

The **decomposition tree** $\mathbf{T}_A$ can be defined as follows

It has the formula $A$ as a **root**

For each **node**, if there is a **rule** of **RS2** which conclusion has the same form as **node** sequence, i.e.

if there is a **decomposition rule** to be applied, then the **node** has children that are premises of the **rule**

## **RS2** Exercises

If the **node** consists only of literals (i.e. there is no decomposition rule to be applied), then it **does not** have any children

The last statement defines a termination condition for the **tree**

This definition **correctly** defines a decomposition tree as it identifies and uses appropriate the **decomposition** rules

## **RS2** Exercises

Since in **RS2** **all** rules of inference have a sequence $\Gamma$ instead of $\Gamma'$ as it was defined for in **RS**, the **choice** of the decomposition rule for a node may be **not unique**

For **example** consider a **node**

$$(a => b), (b \cup a)$$

$\Gamma$ in the **RS2** rules is a sequence of formulas, not literals, so for this **node** we can choose either rule $(=>)$ or rule $(\cup)$ as a **decomposition rule**

This leads to existence of non-unique **trees**

# RS2 Exercises

**Exercise 4**

Prove the **Completeness Theorem** for **RS2**

**Solution**

We need to prove the completeness part only, as the soundness has been already proved, i.e. we have to prove the implication: for any formula A ,

$$\text{if} \quad \nvdash_{RS2} A \quad \text{then} \quad \not\models A$$

**Assume** $\nvdash_{RS2} A$ ,

Then **every** decomposition tree of A has at least one non-axiom **leaf**

Otherwise, there **would exist** a tree with all axiom leaves and it would be a **proof** for A

# **RS2** Exercises

Let $\mathcal{T}_A$ be a set of **all** decomposition trees of $A$

**We choose** an arbitrary $T_A \in \mathcal{T}_A$ with at least one non-axiom leaf $L_A$

The non-axiom leaf $L_A$ **defines** a truth assignment $v$ which falsifies $A$, as follows:

$$v(a) = \begin{cases} F & \text{if } a \text{ appears in } L_A \\ T & \text{if } \neg a \text{ appears in } L_A \\ \text{any value} & \text{if } a \text{ does not appear in } L_A \end{cases}$$

The value for a sequence that corresponds to the leaf in is F

Since, because of the **strong soundness** F ”climbs” the tree, we found a **counter-model** for A, i.e.

$$\not\models A$$

# **RS2** Exercises

**Exercise 5** Write a procedure $TREE_A$ such that for any formula $A$ of **RS2** it produces its **unique** decomposition tree

```
Procedure $TREE_A$(Formula A, Tree T)
{
    $B = ChoseLeftMostFormula(A)$ // Choose the left most formula that is not a literal
    $c = MainConnective(B)$ // Find the main connective of B
    $R = FindRule(c)$// Find the rule which conclusion that has this connective
    $P = Premises(R)$// Get the premises for this rule
    $AddToTree(A, P)$// add premises as children of A to the tree
    For all p in P // go through all premises
        $TREE_A(p, T)$ // build subtrees for each premiss
}
```

# **RS2** Exercises

**Exercise 6**

Prove **completeness** of your **Procedure** $TREE_A$

**Procedure** $TREE_A$ provides a unique tree, since it always chooses the most left indecomposable formula for a choice of a decomposition rule and there is only one such rule

This procedure is equivalent to RS system, since with the decomposition rules of RS the most left decomposable formula is always chosen

RS system is **complete**, thus this Procedure is **complete**