CSE581
Computer Science Fundamentals: Theory

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LECTURE 5

Chapter 5
HILBERT PROOF SYSTEMS: Completeness of Classical Propositional Logic

PART 1: Hilbert Proof System H_1 , **Deduction Theorem** and examples of **formal proofs**

PART 2: System H_2 and **Completeness Theorem** for Classical Propositional Logic

PART 3: Examples of **Complete Proof Systems** for Classical Propositional Logic

Hilbert Proof Systems

Hilbert proof systems are based on a language with **implication** and **contain Modus Ponens** as a rule of inference

Modus Ponens is probably the **oldest** of all known rules of inference as it was already known to the Stoics in 3rd century B.C. and is also considered as the **most natural** to our intuitive thinking

The proof systems containing **Modus Ponens** as the inference rule play a special role in logic.

Hilbert Proof Systems

Hilbert systems put major emphasis on logical axioms and keep the number of **rules** of inference at the minimum
Hilbert systems often **admit** the Modus Ponens as the **sole rule** of inference

There are many proof systems that describe classical propositional logic, i.e. that are **complete** with respect to the classical semantics

We present a **Hilbert** proof system for the classical propositional logic and discuss **two ways** of proving the **Completeness Theorem** for it

Hilbert Proof Systems

The **first proof** is based on the one included in Elliott Mendelson's book Introduction to Mathematical Logic

It is is a **constructive** proof that shows how one can use the assumption that a formula A is a tautology in order to construct its formal **proof**

The **second proof** is **non-constructive**

Its importance lies in a fact that the methods it uses can be applied to the proof of **completeness** for classical predicate logic (chapter 9)

It also **generalizes** to some non-classical logics

We prove completeness part of the **Completeness Theorem** by proving the converse implication to it

We show how one can **deduce** that a formula A **is not** a tautology **from** the fact that it **does not** have a proof

It is hence called a **counter-model** construction proof

Both proofs relay on the **Deduction Theorem** and so this is the first theorem we are now going to prove

Hilbert Proof System H_1

We consider now a **Hilbert** proof system H_1 **based** on a this is language with implication as the **only** connective, with **two** logical axioms, and with Modus Ponens as a **sole rule** of inference

Hilbert Proof System H_1

We define Hilbert system H_1 as follows

$$H_1 = (\ \mathcal{L}_{\{\Rightarrow\}},\ \ \mathcal{F},\ \ \{A1, A2\},\ \ MP\ )$$

A1 (Law of simplification)

$$(A \Rightarrow (B \Rightarrow A))$$

A2 (Frege's Law)

$$((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$$

MP is the Modus Ponens rule

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

where A, B, C are any formulas from $\mathcal{F}$

Formal Proofs in H_1

Finding **formal proofs** in this system requires some ingenuity. **The formal proof** of $(A \Rightarrow A)$ in H_1 is a sequence

$$B_1, \; B_2, \; B_3, \; B_4, B_5$$

as defined below.

B_1 :$((A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)))$,
axiom A2 for $A = A$, $B = (A \Rightarrow A)$, and $C = A$

B_2 :$(A \Rightarrow ((A \Rightarrow A) \Rightarrow A))$,
axiom A1 for $A = A$, $B = (A \Rightarrow A)$

B_3 :$((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)))$,
MP application to B_1 and B_2

B_4 :$(A \Rightarrow (A \Rightarrow A))$,
axiom A1 for $A = A$, $B = A$

B_5 :$(A \Rightarrow A)$
MP application to B_3 and B_4

Searching for Proofs in a Proof System

A **general procedure** for automated search for proofs in a proof system S can be stated is as follows

Let B be an expression of the system *S* that is not an axiom

If B has a **proof** in S, B must be the **conclusion** of one of the inference rules

Let's say it is a rule r

We **find** all its premisses, i.e. we evaluate $r^{-1}(B)$

If **all premisses** are axioms, the proof is **found**

Otherwise we **repeat** the procedure for any **premiss** that **is not** an axiom

Search for Proof by the Means of MP

The MP rule says:

given two formulas A and $(A \Rightarrow B)$ we conclude a formula B

Assume now that and want to find a **proof** of a formula B

If B is an **axiom**, we have the **proof**; the formula itself

If B **is not** an axiom, it had to be obtained by the application of the Modus Ponens rule to certain two formulas A and $(A \Rightarrow B)$ and there is **infinitely many** of such formulas!

The proof system H_1 **is not** syntactically decidable

Semantic Links

Semantic Link 1

System H_1 is **sound** under classical semantics and H_1 is **not sound** under **K** semantics

Soundness Theorem for H_1

For any $A \in \mathcal{F}$, **if** $\vdash_{H_1} A$, **then** $\models A$

Semantic Links

Semantic Link 2

The system H_1 **is not complete** under classical semantics

Not all classical **tautologies** have a proof in H_1

We proved that can't define **negation** in term of implication alone and so for example, a basic **tautology** $(\neg\neg A \Rightarrow A)$ is not provable in H_1, i.e.

$$\nvdash_{H_1} (\neg\neg A \Rightarrow A)$$

Proof from Hypothesis

Given a proof system $S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$

While proving expressions we often use some extra information available, besides the axioms of the proof system

This extra information is called **hypothesis** in the proof

Let $\Gamma \subseteq \mathcal{E}$ be any set expressions called hypothesis

We **write** $\Gamma \vdash_S E$ to **denote** that

" E has a proof in S from the set Γ and the logical axioms LA"

Formal Definition

Definition

We say that $E \in \mathcal{E}$ has a **formal proof** in S from the set Γ and the logical axioms LA and denote it as $\Gamma \vdash_S E$ if and only if there is a sequence

$$A_1, \ldots, A_n$$

of expressions from $\mathcal{E}$, such that

$$A_1 \in LA \cup \Gamma, \quad A_n = E$$

and for each $1 < i \leq n$, either $A_i \in LA \cup \Gamma$ or A_i is a **direct consequence** of some of the **preceding** expressions by virtue of one of the rules of inference of S

Deduction Theorem for H_1

Deduction Theorem for H_1

For any $A, B \in \mathcal{F}$ and $\Gamma \subseteq \mathcal{F}$

$$\Gamma,\ A \vdash_{H_1} B \quad \text{if and only if} \quad \Gamma \vdash_{H_1} (A \Rightarrow B)$$

In particular

$$A \vdash_{H_1} B \quad \text{if and only if} \quad \vdash_{H_1} (A \Rightarrow B)$$

The proof of the following **Lemma** provides a good example of multiple applications of the **Deduction Theorem**

Lemma

For any $A, B, C \in \mathcal{F}$,

(a) $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)$,

(b) $(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$

Observe that by **Deduction Theorem** we can re-write **(a)** as

(a') $(A \Rightarrow B),\ (B \Rightarrow C), A \vdash_{H_1} C$

Poof of **(a')**

We construct a formal proof

$$B_1, B_2, B_3, B_4, B_5$$

of $(A \Rightarrow B), (B \Rightarrow C), A \vdash_{H_1} C$ as follows.

B_1 : $(A \Rightarrow B)$

hypothesis

B_2 : $(B \Rightarrow C)$

hypothesis

B_3 : A

hypothesis

B_4 : B

B_1, B_3 and MP

B_5 : C

B_2, B_4 and MP

Formal Proofs

Thus we proved by **Deduction Theorem** that **(a)** holds, i.e.

$$(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)$$

Proof of **Lemma** part **(b)**

By **Deduction Theorem** we have that

$$(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$$

if and only if

$$(A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C)$$

Formal Proof

Here is a simple proof of **Lemma** part **(b)**

We apply the **Deduction Theorem** twice, i.e. we get

$$(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$$

if and only if

$$(A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C)$$

if and only if

$$(A \Rightarrow (B \Rightarrow C)), B, A \vdash_{H_1} C$$

Simple Proof

We now construct a proof of $(A \Rightarrow (B \Rightarrow C)), B, A \vdash_{H_1} C$ as follows

B_1 : $(A \Rightarrow (B \Rightarrow C))$
hypothesis

B_2 : B
hypothesis

B_3 : A
hypothesis

B_4 : $(B \Rightarrow C)$
B_1 , B_3 and (MP)

B_5 : C
B_2 , B_4 and (MP)

Classical Propositional Proof System H_2

Hilbert System H_2

The proof system H_1 is **sound** and strong enough to prove the Deduction Theorem, but it is **not complete**

We extend now its language and the set of logical axioms to a **complete set of axioms**

We define a system H_2 that is **complete** with respect to the classical semantics

The proof of completeness theorem is be presented in the next chapter.

Definition

$$H_2 = (\ \mathcal{L}_{\{\Rightarrow,\neg\}},\ \mathcal{F},\ \{A1, A2, A3\}\ \ (MP)\ )$$

A1 (Law of simplification)
$(A \Rightarrow (B \Rightarrow A))$

A2 (Frege's Law)
$((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

MP (Rule of inference)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

where A, B, C are any formulas of the propositional language $\mathcal{L}_{\{\Rightarrow,\neg\}}$

Deduction Theorem for System H_2

Observation 1

The proof system H_2 is obtained by adding axiom A_3 to the system H_1

Observation 2

The language of H_2 is obtained by adding the connective $\neg$ to the language of H_1

Observation 3

The use of axioms $A1, A2$ in the proof of **Deduction Theorem** for the system H_1 is independent of the connective $\neg$ added to the language of H_1

Observation 4

Hence the proof of the **Deduction Theorem** for the system H_1 can be repeated **as it is** for the system H_2

Deduction Theorem for System H_2

Observations 1-4 prove that he Deduction Theorem holds for system H_2

Deduction Theorem for H_2

For any $\Gamma \subseteq \mathcal{F}$ and $A, B \in \mathcal{F}$

$$\Gamma,\ A \vdash_{H_2} B \text{ if and only if } \Gamma \vdash_{H_2} (A \Rightarrow B)$$

In particular

$$A \vdash_{H_2} B \text{ if and only if } \vdash_{H_2} (A \Rightarrow B)$$

Soundness and CompletenessTheorems

We get by easy verification

Soundness Theorem H_2

For every formula $A \in \mathcal{F}$

$$\text{if } \vdash_{H_2} A \quad \text{then} \quad \models A$$

We prove in the next Lecture, that H_2 is also complete, i.e. we prove

Completeness Theorem for H_2

For every formula $A \in \mathcal{F}$,

$$\vdash_{H_2} A \ \text{ if and only if } \ \models A$$

CompletenessTheorems

The proof of completeness theorem (for a given semantics) is always a **main point** in creation of any new logic

There are many techniques to prove it, depending on the proof system, and on the semantics we define for it

We **present** in Lecture 5a and Lecture 5b two proofs of the **Completeness Theorem** for the system H_2

These proofs use very different techniques, hence the **reason** of presenting both of them

Hilbert Proof Systems
Completeness of Classical Propositional Logic

PART 3: Some other **Complete Axiomatizations** for Classical Propositional Logic

Some Other Axiomatizations

We present here some of the most **known**, and historically important **axiomatizations** of classical propositional logic

It means the **Hilbert** proof systems that are proven to be **complete** under classical semantics

Lukasiewicz

Lukasiewicz (1929)

The Lukasiewicz proof system (axiomatization) is

$$L = (\ \mathcal{L}_{\{\neg,\ \Rightarrow\}},\ \ \mathcal{F},\ \ A1, A2, A3,\ \ MP\ )$$

where

A1 $((\neg A \Rightarrow A) \Rightarrow A)$

A2 $(A \Rightarrow (\neg A \Rightarrow B))$

A3 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$

for any formulas $A, B, C \in \mathcal{F}$

Hilbert and Ackermann (1928)

$$HA = (\ \mathcal{L}_{\{\neg, \cup\}},\ \ \mathcal{F},\ \ A1 - A4,\ \ MP\ )$$

where for any $A, B, C \in \mathcal{F}$

A1 $(\neg(A \cup A) \cup A)$

A2 $(\neg A \cup (A \cup B))$

A3 $(\neg(A \cup B) \cup (B \cup A))$

A4 $(\neg(\neg B \cup C) \cup (\neg(A \cup B) \cup (A \cup C)))$

The Modus Ponens rule in the language $\mathcal{L}_{\{\neg, \cup\}}$ has a form

$$MP\ \ \frac{A\ ;\ \ (\neg A \cup B)}{B}$$

Observe that also the **Deduction Theorem** is now formulated as follow.

Deduction Theorem for *HA*

For any subset Γ of the set of formulas $\mathcal{F}$ of HA and for any formulas $A, B \in \mathcal{F}$,

$$\Gamma,\ A \vdash_{HA} B \quad \text{if and only if} \quad \Gamma \vdash_{HA} (\neg A \cup B)$$

In particular,

$$A \vdash_{HA} B \quad \text{if and only if} \quad \vdash_{HA} (\neg A \cup B)$$

Hilbert (1928)

$$H = (\ \mathcal{L}_{\{\neg, \cup, \cap, \Rightarrow\}},\ \ \mathcal{F},\ \ A1 - A15,\ \ MP\ )$$

where for any $A, B, C \in \mathcal{F}$

A1 $(A \Rightarrow A)$

A2 $(A \Rightarrow (B \Rightarrow A))$

A3 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$

A4 $((A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B))$

A5 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C)))$

A6 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$

A7 $((A \cap B) \Rightarrow A)$

A8 $((A \cap B) \Rightarrow B)$

Hilbert

A9 $((A \Rightarrow B) \Rightarrow ((A \Rightarrow C) \Rightarrow (A \Rightarrow (B \cap C)))$

A10 $(A \Rightarrow (A \cup B))$

A11 $(B \Rightarrow (A \cup B))$

A12 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$

A13 $((A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A))$

A14 $(\neg A \Rightarrow (A \Rightarrow B))$

A1 - A14 are the axioms **Hilbert** proposed and were accepted as axioms defining Intuitionistic logic

They were later **proved** to be **complete** when the intuitionistic **semantics** was discovered

Hilbert obtained his classical axiomatization by adding as the last axiom the **excluded middle** law rejected by intuitionists

A15 $(A \cup \neg A)$

Kleene (1952)

$$K = (\ \mathcal{L}_{\{\neg,\cup,\cap,\Rightarrow\}},\ \ \mathcal{F},\ \ A1 - A10,\ \ MP\ )$$

where for any $A, B, C \in \mathcal{F}$

A1 $(A \Rightarrow (B \Rightarrow A))$

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C)))$

A3 $((A \cap B) \Rightarrow A)$

A4 $((A \cap B) \Rightarrow B)$

A5 $(A \Rightarrow (B \Rightarrow (A \cap B)))$

Kleene

A6 $(A \Rightarrow (A \cup B))$

A7 $(B \Rightarrow (A \cup B))$

A8 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$

A9 $((A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A))$

A10 $(\neg\neg A \Rightarrow A)$

Kleene proved that when A10 is **replaced** by

A10' $(\neg A \Rightarrow (A \Rightarrow B))$

the **resulting** system is a **complete** axiomatization of Intuitionistic Logic

Rasiowa-Sikorski

Rasiowa-Sikorski (1950)

$$RS = (\ \mathcal{L}_{\{\neg,\cup,\cap,\Rightarrow\}},\ \ \mathcal{F},\ \ A1 - A12,\ \ MP\ )$$

where for any $A, B, C \in \mathcal{F}$

A1 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$

A2 $(A \Rightarrow (A \cup B))$

A3 $(B \Rightarrow (A \cup B))$

A4 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$

Rasiowa-Sikorski

A5 $((A \cap B) \Rightarrow A)$

A6 $((A \cap B) \Rightarrow B)$

A7 $((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B)))$

A8 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))$

A9 $(((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$

A10 $(A \cap \neg A) \Rightarrow B)$

A11 $((A \Rightarrow (A \cap \neg A)) \Rightarrow \neg A)$

A12 $(A \cup \neg A)$

Rasiowa-Sikorski

Rasiowa - Sikorski proved A1 - A11 to be a **complete** axiomatization for the Intuitionistic Logic

They obtained the classical axiomatization by adding A12, the **excluded middle** law rejected by intuitionists, as **Hilbert** did

Both classical and intuitionistic **completeness** proofs were carried under respective Boolean and Pseudo-Boolean algebras semantics what is reflected in the **choice** of axioms A1 - A12

Shortest Axiomatizations

Here is the shortest axiomatization for the language

$$\mathcal{L}_{\{\neg,\ \Rightarrow\}}$$

It contains just one axiom

Meredith (1953)

$$M = (\ \mathcal{L}_{\{\neg,\ \Rightarrow\}},\ \ \mathcal{F},\ \ A1\ \ MP\ )$$

where

A1 $((((((A \Rightarrow B) \Rightarrow (\neg C \Rightarrow \neg D)) \Rightarrow C) \Rightarrow E)) \Rightarrow ((E \Rightarrow A) \Rightarrow (D \Rightarrow A)))$

Here is another axiomatization that uses only one axiom

Jean Nicod (1917)

$$N = (\ \mathcal{L}_{\{\uparrow\}},\ \mathcal{F},\ A1,\ (r)\ )$$

where

A1 $(((A \uparrow (B \uparrow C)) \uparrow ((D \uparrow (D \uparrow D)) \uparrow ((E \uparrow B) \uparrow ((A \uparrow E) \uparrow (A \uparrow E))))))$

and

$$(r)\ \frac{A \uparrow (B \uparrow C)}{A}$$

Reminder

We have proved in chapter 3 that

$$\mathcal{L}_{\{\neg,\cup,\cap,\Rightarrow\}} \equiv \mathcal{L}_{\{\uparrow\}}$$