CSE581
Computer Science Fundamentals: Theory

Professor Anita Wasilewska

LECTURE 5

Chapter 5
HILBERT PROOF SYSTEMS: Completeness of Classical Propositional Logic

PART 1: Hilbert Proof System H_1 and examples of applications of Deduction Theorem

PART 2: Proof of Deduction Theorem for System H_1

PART 3: System H_2 and examples of **formal proofs** in H_2

Hilbert Proof Systems

Hilbert proof systems are based on a language with **implication** and **contain Modus Ponens** as a rule of inference

Modus Ponens is probably the **oldest** of all known rules of inference as it was already known to the Stoics in 3rd century B.C. and is also considered as the **most natural** to our intuitive thinking

The proof systems containing **Modus Ponens** as the inference rule play a special role in logic.

Hilbert Proof Systems

Hilbert systems put major emphasis on logical axioms and keep the number of **rules** of inference at the minimum
Hilbert systems often **admit** the Modus Ponens as the **sole rule** of inference

There are many proof systems that describe classical propositional logic, i.e. that are **complete** with respect to the classical semantics

We present a **Hilbert** proof system for the classical propositional logic and discuss **two ways** of proving the **Completeness Theorem** for it

Hilbert Proof Systems

The **first proof** is based on the one included in Elliott Mendelson's book Introduction to Mathematical Logic

It is is a **constructive** proof that shows how one can use the assumption that a formula A is a tautology in order to construct its formal **proof**

The **second proof** is **non-constructive**

Its importance lies in a fact that the methods it uses can be applied to the proof of **completeness** for classical predicate logic (chapter 9)

It also **generalizes** to some non-classical logics

Hilbert Proof Systems

We prove completeness part of the **Completeness Theorem** by proving the converse implication to it

We show how one can **deduce** that a formula A **is not** a tautology **from** the fact that it **does not** have a proof

It is hence called a **counter-model** construction proof

Both proofs relay on the **Deduction Theorem** and so this is the first theorem we are now going to prove

Hilbert Proof System H_1

We consider now a **Hilbert** proof system H_1 **based** on a this is language with implication as the **only** connective, with **two** logical axioms, and with Modus Ponens as a **sole rule** of inference

Hilbert Proof System H_1

We define Hilbert system H_1 as follows

$$H_1 = (\ \mathcal{L}_{\{\Rightarrow\}},\ \ \mathcal{F},\ \ \{A1, A2\},\ \ MP\ )$$

A1 (Law of simplification)

$$(A \Rightarrow (B \Rightarrow A))$$

A2 (Frege's Law)

$$((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$$

MP is the Modus Ponens rule

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

where A, B, C are any formulas from $\mathcal{F}$

Formal Proofs in H_1

Finding **formal proofs** in this system requires some ingenuity.
The formal proof of $(A \Rightarrow A)$ in H_1 is a sequence

$$B_1,\ B_2,\ B_3,\ B_4, B_5$$

as defined below.

B_1 :$((A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)))$,
axiom A2 for $A = A$, $B = (A \Rightarrow A)$, and $C = A$

B_2 :$(A \Rightarrow ((A \Rightarrow A) \Rightarrow A))$,
axiom A1 for $A = A$, $B = (A \Rightarrow A)$

B_3 :$((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)))$,
MP application to B_1 and B_2

B_4 :$(A \Rightarrow (A \Rightarrow A))$,
axiom A1 for $A = A$, $B = A$

B_5 :$(A \Rightarrow A)$
MP application to B_3 and B_4

Searching for Proofs in a Proof System

A **general procedure** for automated search for proofs in a proof system S can be stated is as follows

Let B be an expression of the system S that is not an axiom

If B has a **proof** in S, B must be the **conclusion** of one of the inference rules

Let's say it is a rule r

We **find** all its premisses, i.e. we evaluate $r^{-1}(B)$

If **all premisses** are axioms, the proof is **found**

Otherwise we **repeat** the procedure for any **premiss** that **is not** an axiom

Search for Proof by the Means of MP

The MP rule says:

given two formulas A and $(A \Rightarrow B)$ we conclude a formula B

Assume now that and want to find a **proof** of a formula B

If B is an **axiom**, we have the **proof**; the formula itself

If B **is not** an axiom, it had to be obtained by the application of the Modus Ponens rule to certain two formulas A and $(A \Rightarrow B)$ and there is **infinitely many** of such formulas!

The proof system H_1 **is not** syntactically decidable

Semantic Links

Semantic Link 1

System H_1 is **sound** under classical semantics and H_1 is **not sound** under **K** semantics

Soundness Theorem for H_1

For any $A \in \mathcal{F}$, **if** $\vdash_{H_1} A$, **then** $\models A$

Semantic Links

Semantic Link 2

The system H_1 **is not complete** under classical semantics

Not all classical **tautologies** have a proof in H_1

We proved that can't define **negation** in term of implication alone and so for example, a basic **tautology** $(\neg\neg A \Rightarrow A)$ is not provable in H_1 , i.e.

$$\nvdash_{H_1} (\neg\neg A \Rightarrow A)$$

Proof from Hypothesis

Given a proof system $S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$

While proving expressions we often use some extra information available, besides the axioms of the proof system

This extra information is called **hypothesis** in the proof

Let $\Gamma \subseteq \mathcal{E}$ be any set expressions called hypothesis

We **write** $\Gamma \vdash_S E$ to **denote** that

" E has a proof in S from the set Γ and the logical axioms LA"

Formal Definition

Definition

We say that $E \in \mathcal{E}$ has a **formal proof** in S from the set Γ and the logical axioms LA and denote it as $\Gamma \vdash_S E$ if and only if there is a sequence

$$A_1, \ldots, A_n$$

of expressions from $\mathcal{E}$, such that

$$A_1 \in LA \cup \Gamma, \quad A_n = E$$

and for each $1 < i \leq n$, either $A_i \in LA \cup \Gamma$ or A_i is a **direct consequence** of some of the **preceding** expressions by virtue of one of the rules of inference of S

Special Cases

Case 1: $\Gamma \subseteq \mathcal{E}$ is a **finite set** and $\Gamma = \{B_1, B_2, ..., B_n\}$

We write

$$B_1, B_2, ..., B_n \vdash_{\mathcal{S}} E$$

instead of $\{B_1, B_2, ..., B_n\} \vdash_{\mathcal{S}} E$

Case 2: $\Gamma = \emptyset$

By the **definition** of a proof of E from Γ, $\emptyset \vdash_S E$ means that in the proof of E we use **only** the logical axioms LA of S

We hence write

$$\vdash_S E$$

to denote that E has a proof from $\Gamma = \emptyset$

Proof from Hypothesis in H_1

Show that

$$(A \Rightarrow B),\ (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)$$

We construct a **formal proof**

$$B_1, B_2, ...B_7$$

of $(A \Rightarrow C)$ from hypothesis $(A \Rightarrow B)$ and $(B \Rightarrow C)$ as follows

Proof from Hypothesis in H_1

$B_1 : (B \Rightarrow C)$, $B_2 : (A \Rightarrow B)$,
hypothesis hypothesis

$B_3 : ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,
axiom A2

$B_4 : ((B \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)))$,
axiom A1 for $A = (B \Rightarrow C)$, $B = A$

$B_5 : (A \Rightarrow (B \Rightarrow C))$,
B_1 and B_4 and MP

$B_6 : ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$, $B_7 : (A \Rightarrow C)$
MP

Deduction Theorem

In mathematical arguments, one often **proves** a statement B on the assumption of some other statement A and then concludes that we have **proved** the implication "if A, then B"

This reasoning is justified by a following theorem, called a **Deduction Theorem**

Reminder

We write $\Gamma, A \vdash B$ for $\Gamma \cup \{A\} \vdash B$

In general, we write $\Gamma, A_1, A_2, ..., A_n \vdash B$

for $\Gamma \cup \{A_1, A_2, ..., A_n\} \vdash B$

Deduction Theorem for H_1

Deduction Theorem for H_1

For any $A, B \in \mathcal{F}$ and $\Gamma \subseteq \mathcal{F}$

$$\Gamma,\ A \vdash_{H_1} B \quad \text{if and only if} \quad \Gamma \vdash_{H_1} (A \Rightarrow B)$$

In particular

$$A \vdash_{H_1} B \quad \text{if and only if} \quad \vdash_{H_1} (A \Rightarrow B)$$

The proof of the following **Lemma** provides a good example of multiple applications of the **Deduction Theorem**

Lemma

For any $A, B, C \in \mathcal{F}$,

(a) $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)$,

(b) $(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$

Observe that by **Deduction Theorem** we can re-write **(a)** as

(a') $(A \Rightarrow B),\ (B \Rightarrow C), A \vdash_{H_1} C$

Poof of **(a')**

We construct a formal proof

$$B_1, B_2, B_3, B_4, B_5$$

of $(A \Rightarrow B), (B \Rightarrow C), A \vdash_{H_1} C$ as follows.

B_1 : $(A \Rightarrow B)$
hypothesis

B_2 : $(B \Rightarrow C)$
hypothesis

B_3 : A
hypothesis

B_4 : B
B_1, B_3 and MP

B_5 : C
B_2, B_4 and MP

Thus we proved by **Deduction Theorem** that **(a)** holds, i.e.

$$(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)$$

Proof of **Lemma** part **(b)**

By **Deduction Theorem** we have that

$$(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$$

if and only if

$$(A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C)$$

Formal Proofs

We construct a formal proof

$$B_1, B_2, B_3, B_4, B_5, B_6, B_7$$

of $(A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C)$ as follows.

B_1 : $(A \Rightarrow (B \Rightarrow C))$
hypothesis

B_2 : B
hypothesis

B_3 : $((B \Rightarrow (A \Rightarrow B))$
$A1$ for $A = B, B = A$

B_4 : $(A \Rightarrow B)$
B_2, B_3 and MP

Formal Proofs

B_5 : $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$
axiom A2

B_6 : $((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$
B_1, B_5 and MP

B_7 : $(A \Rightarrow C)$

Thus we proved by **Deduction Theorem** that

$$(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$$

Simpler Proof

Here i a simpler proof of **Lemma** part **(b)**

We apply the **Deduction Theorem** twice, i.e. we get

$$(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$$

if and only if

$$(A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C)$$

if and only if

$$(A \Rightarrow (B \Rightarrow C)), B, A \vdash_{H_1} C$$

Simpler Proof

We now construct a proof of $(A \Rightarrow (B \Rightarrow C)), B, A \vdash_{H_1} C$ as follows

B_1 : $(A \Rightarrow (B \Rightarrow C))$
hypothesis

B_2 : B
hypothesis

B_3 : A
hypothesis

B_4 : $(B \Rightarrow C)$
B_1 , B_3 and (MP)

B_5 : C
B_2 , B_4 and (MP)

CONSEQUENCE OPERATION

Review

Definition: Consequences of Γ

Given a proof system

$$S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$$

For any $\Gamma \subseteq \mathcal{E}$, and $A \in \mathcal{E}$,

If $\Gamma \vdash_S A$, then A is called a **consequence** of Γ in S

We denote by $\mathbf{Cn}_S(\Gamma)$ the **set of all consequences** of Γ in S, i.e. we put

$$\mathbf{Cn}_S(\Gamma) = \{A \in \mathcal{E} : \quad \Gamma \vdash_S A\}$$

Definition: Consequence Operation

Observe that by defining a consequence of Γ in S, we define in fact a **function** which to every set $\Gamma \subseteq \mathcal{E}$ assigns a set of **all its consequences** $\mathbf{Cn}_S(\Gamma)$

We denote this function by $\mathbf{Cn}_S$ and adopt the following

Definition

Any function

$$\mathbf{Cn}_S : 2^{\mathcal{E}} \longrightarrow 2^{\mathcal{E}}$$

such that for every $\Gamma \in 2^{\mathcal{E}}$

$$\mathbf{Cn}_S(\Gamma) = \{A \in \mathcal{E} : \ \Gamma \vdash_S A\}$$

is called the **consequence operation determined** by S

Consequence Operation: Monotonicity

Take any **consequence operation** determined by S

$$\mathbf{Cn}_S : 2^{\mathcal{E}} \longrightarrow 2^{\mathcal{E}}$$

Monotonicity Property

For any sets Γ, Δ of expressions of S,

if $\Gamma \subseteq \Delta$ **then** $\mathbf{Cn}_S(\Gamma) \subseteq \mathbf{Cn}_S(\Delta)$

Exercise: write the proof;

it follows directly from the definition of $\mathbf{Cn}_S$ and definition of the formal proof

Consequence Operation: Transitivity

Take any **consequence operation**

$$\mathbf{Cn}_S : 2^{\mathcal{E}} \longrightarrow 2^{\mathcal{E}}$$

Transitivity Property

For any sets $\Gamma_1, \Gamma_2, \Gamma_3$ of expressions of S,

if $\Gamma_1 \subseteq \mathbf{Cn}_S(\Gamma_2)$ and $\Gamma_2 \subseteq \mathbf{Cn}_S(\Gamma_3)$, **then** $\Gamma_1 \subseteq \mathbf{Cn}_S(\Gamma_3)$

Exercise: write the proof;

it follows directly from the definition of $\mathbf{Cn}_S$ and definition of the formal proof

Consequence Operation: Finiteness

Take any **consequence operation** determined by

$$\mathbf{Cn}_S : 2^{\mathcal{E}} \longrightarrow 2^{\mathcal{E}}$$

Finiteness Property

For any expression $A \in \mathcal{E}$ and any set $\Gamma \subseteq \mathcal{E}$,
$A \in \mathbf{Cn}_S(\Gamma)$ if and only if there is a **finite subset** Γ_0 of Γ such that $A \in \mathbf{Cn}_S(\Gamma_0)$

Exercise: write the proof;

it follows directly from the definition of $\mathbf{Cn}_S$ and definition of the formal proof

Proof Deduction Theorem for H_1

The Deduction Theorem

As we now fix the proof system to be H_1, we write $A \vdash B$ instead of $A \vdash_{H_1} B$

Deduction Theorem (Herbrand, 1930) for H_1

For any formulas $A, B \in \mathcal{F}$,

$$\text{If } A \vdash B, \quad \text{then } \vdash (A \Rightarrow B)$$

Deduction Theorem (General case) for H_1

For any formulas $A, B \in \mathcal{F}$, $\Gamma \subseteq \mathcal{F}$

$$\Gamma,\ A \vdash B \quad \text{if and only if} \quad \Gamma \vdash (A \Rightarrow B)$$

Proof:

Part 1 We first prove the "if" part:

$$\text{If } \Gamma,\ A \vdash B \quad \text{then} \quad \Gamma \vdash (A \Rightarrow B)$$

Proof of The Deduction Theorem

Assume that

$$\Gamma,\ A\ \vdash B$$

i.e. that we have a formal proof

$$B_1, B_2, ..., B_n$$

of B from the set of formulas $\Gamma \cup \{A\}$

We have to show that

$$\Gamma \vdash (A \Rightarrow B)$$

Proof of The Deduction Theorem

In order to prove that

$\Gamma \vdash (A \Rightarrow B)$ follows from $\Gamma, A \vdash B$

we prove a **stronger statement**, namely that

$$\Gamma \vdash (A \Rightarrow B_i)$$

for any B_i, $1 \leq i \leq n$ in the formal proof $B_1, B_2, ..., B_n$ of B also follows from $\Gamma, A \vdash B$

Hence in **particular case**, when $i = n$ we will obtain that $\Gamma \vdash (A \Rightarrow B)$ follows from $\Gamma, A \vdash B$

and that will end the proof of **Part 1**

Base Step

The proof of **Part 1** is conducted by **mathematical induction** on i, for $1 \leq i \leq n$

Step 1 $i = 1$ (base step)

Observe that when $i = 1$, it means that the **formal proof** $B_1, B_2, ..., B_n$ contains only **one element** B_1

By the **definition** of the formal proof from $\Gamma \cup \{A\}$, we have that

(1) B_1 is a logical axiom, or $B_1 \in \Gamma$, or

(2) $B_1 = A$

This means that $B_1 \in \{A1, A2\} \cup \Gamma \cup \{A\}$

Base Step

Now we have **two cases** to consider.

Case1: $B_1 \in \{A1, A2\} \cup \Gamma$

Observe that $(B_1 \Rightarrow (A \Rightarrow B_1))$ is the axiom $A1$

By assumption $B_1 \in \{A1, A2\} \cup \Gamma$

We get the **required proof** of $(A \Rightarrow B_1)$ from Γ

by the following application of the Modus Ponens rule

$$(MP)\ \frac{B_1\ ;\ (B_1 \Rightarrow (A \Rightarrow B_1))}{(A \Rightarrow B_1)}$$

Base Step

Case 2: $B_1 = A$

When $B_1 = A$ then to prove $\Gamma \vdash (A \Rightarrow B_1)$

This means we have to prove

$$\Gamma \vdash (A \Rightarrow A)$$

This holds by **monotonicity** of the consequence and the fact that we have shown that

$$\vdash (A \Rightarrow A)$$

The above cases **conclude the proof** for $i = 1$ of

$$\Gamma \vdash (A \Rightarrow B_i)$$

Inductive Step

Inductive Step

Assume that

$$\Gamma \vdash (A \Rightarrow B_k)$$

for all $k < i$ (strong induction)

We will **show** that using this fact we can conclude that also

$$\Gamma \vdash (A \Rightarrow B_i)$$

Inductive Step

Consider a formula B_i in the formal proof

$$B_1, B_2, ..., B_n$$

By **definition** of the formal proof we have to show the following tow cases

Case 1 : $B_i \in \{A1, A2\} \cup \Gamma \cup \{A\}$ and

Case 2: B_i follows by MP from certain B_j, B_m such that $j < m < i$

Consider now the **Case 1:** $B_i \in \{A1, A2\} \cup \Gamma \cup \{A\}$

The proof of $(A \Rightarrow B_i)$

from Γ in this case is **obtained** from the proof of the **Step** $i = 1$ by replacement B_1 by B_i

and is omitted here as a **straightforward repetition**

Inductive Step

Case 2:

B_i is a **conclusion** of (MP)

If B_i is a conclusion of (MP), then we must have two formulas B_j, B_m in the formal proof

$$B_1, B_2, ..., B_n$$

such that $j < i$, $m < i$, $j \neq m$ and

$$(MP)\ \frac{B_j\ ;\ B_m}{B_i}$$

Inductive Step

By the **inductive assumption** the formulas B_j, B_m are such that $\Gamma \vdash (A \Rightarrow B_j)$ and $\Gamma \vdash (A \Rightarrow B_m)$

Moreover, by the definition of (MP) rule, the formula B_m has to

have a form $(B_j \Rightarrow B_i)$

This means that

$$B_m = (B_j \Rightarrow B_i)$$

The inductive assumption can be re-written as follows

$$\Gamma \vdash (A \Rightarrow (B_j \Rightarrow B_i))$$

for $j < i$

Inductive Step

Observe now that the formula

$$((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

is a **substitution of the axiom A2** and hence **has a proof** in our system

By the monotonicity of the consequence, it also has a proof from the set Γ, i.e.

$$\Gamma \vdash ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

Inductive Step

We know that

$$\Gamma \vdash ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

Applying the rule MP i.e. performing the following

$$\frac{(A \Rightarrow (B_j \Rightarrow B_i)) \ ; \ ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))}{((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}$$

we get that also

$$\Gamma \vdash ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))$$

Inductive Step

Applying again the rule MP i.e. performing the following

$$\left(\frac{(A \Rightarrow B_j)\ ;\ ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}{(A \Rightarrow B_i)}\right)$$

we get that

$$\Gamma \vdash (A \Rightarrow B_i)$$

what **ends the proof** of the inductive step

Proof of the Deduction Theorem

By the mathematical induction principle, we have **proved** that

$$\Gamma \vdash (A \Rightarrow B_i), \quad \text{for all} \quad 1 \leq i \leq n$$

In particular it is **true** for $i = n$, i.e. for $B_n = B$

and we proved that

$$\Gamma \vdash (A \Rightarrow B)$$

This ends the proof of the **first part** of the **Deduction Theorem**:

$$\text{If} \quad \Gamma, A \vdash B, \quad \text{then} \quad \Gamma \vdash (A \Rightarrow B)$$

Proof of the Deduction Theorem

The **proof** of the second part, i.e. of the inverse implication:

$$\text{If} \quad \Gamma \vdash (A \Rightarrow B), \quad \text{then} \quad \Gamma, A \vdash B$$

is **straightforward** and goes as follows.

Assume that $\Gamma \vdash (A \Rightarrow B)$

By the monotonicity of the consequence we have also that $\Gamma, A \vdash (A \Rightarrow B)$

Obviously $\Gamma, A \vdash A$

Applying Modus Ponens to the above, we get the proof of B from $\{\Gamma, A\}$

We have hence proved that

$$\Gamma, A \vdash B$$

Proof of the Deduction Theorem

This **ends** the proof of

Deduction Theorem (General case) for H_1

For any formulas $A, B \in \mathcal{F}$ and any $\Gamma \subseteq \mathcal{F}$

$$\Gamma,\ A \vdash B \quad \text{if and only if} \quad \Gamma \vdash (A \Rightarrow B)$$

The particular case we get also the particular case

Deduction Theorem (Herbrand, 1930) for H_1

For any formulas $A, B \in \mathcal{F}$,

$$\text{If } \ A \vdash B, \quad \text{then } \ \vdash (A \Rightarrow B)$$

is obtained from the above by assuming that the set Γ is empty

Classical Propositional Proof System H_2

Hilbert System H_2

The proof system H_1 is **sound** and strong enough to prove the Deduction Theorem, but it is **not complete**
We extend now its language and the set of logical axioms to a **complete set of axioms**

We define a system H_2 that is **complete** with respect to the classical semantics

The proof of completeness theorem is be presented in the next chapter.

Hilbert System H_2 Definition

Definition

$$H_2 = (\ \mathcal{L}_{\{\Rightarrow,\neg\}},\ \mathcal{F},\ \{A1, A2, A3\}\ \ (MP)\ )$$

A1 (Law of simplification)
$(A \Rightarrow (B \Rightarrow A))$

A2 (Frege's Law)
$((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

MP (Rule of inference)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

where A, B, C are any formulas of the propositional language $\mathcal{L}_{\{\Rightarrow,\neg\}}$

Deduction Theorem for System H_2

Observation 1

The proof system H_2 is obtained by adding axiom A_3 to the system H_1

Observation 2

The language of H_2 is obtained by adding the connective $\neg$ to the language of H_1

Observation 3

The use of axioms $A1, A2$ in the proof of **Deduction Theorem** for the system H_1 is independent of the connective $\neg$ added to the language of H_1

Observation 4

Hence the proof of the **Deduction Theorem** for the system H_1 can be repeated **as it is** for the system H_2

Deduction Theorem for System H_2

Observations 1-4 prove that he Deduction Theorem holds for system H_2

Deduction Theorem for H_2

For any $\Gamma \subseteq \mathcal{F}$ and $A, B \in \mathcal{F}$

$$\Gamma,\ A \vdash_{H_2} B \quad \text{if and only if} \quad \Gamma \vdash_{H_2} (A \Rightarrow B)$$

In particular

$$A \vdash_{H_2} B \quad \text{if and only if} \quad \vdash_{H_2} (A \Rightarrow B)$$

Soundness and CompletenessTheorems

We get by easy verification

Soundness Theorem H_2

For every formula $A \in \mathcal{F}$

$$\text{if } \vdash_{H_2} A \quad \text{then} \quad \models A$$

We prove in the next Lecture, that H_2 is also complete, i.e. we prove

Completeness Theorem for H_2

For every formula $A \in \mathcal{F}$,

$$\vdash_{H_2} A \text{ if and only if } \models A$$

CompletenessTheorems

The proof of completeness theorem (for a given semantics) is always a **main point** in creation of any new logic

There are many techniques to prove it, depending on the proof system, and on the semantics we define for it

We **present** in Lecture 5a and Lecture 5b two proofs of the **Completeness Theorem** for the system H_2

These proofs use very different techniques, hence the **reason** of presenting both of them

FORMAL PROOFS IN H_2

Examples and Exercises

We present now some examples of **formal proofs** in H_2

There are two reasons for presenting them.

First reason is that all formulas we prove here to be provable play a crucial role in the **proof** of Completeness Theorem for H_2

The second reason is that they provide a "training ground" for a reader to learn how to develop formal proofs

For this reason we write some proofs in a full detail and we leave some for the reader to complete in a way explained in the following example.

Important Lemma

We write $\vdash$ instead of $\vdash_{H_2}$ for the sake of simplicity

Reminder

In the construction of the formal proofs we often use the **Deduction Theorem** and the following **Lemma 1** they was proved in previous section

Lemma 1

$$(\mathbf{a})\quad (A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C)$$

$$(\mathbf{b})\quad (A \Rightarrow (B \Rightarrow C)) \vdash_{H_2} ((B \Rightarrow (A \Rightarrow C))$$

Example 1

Here are consecutive steps

$$B_1, ..., B_5,\ B_6$$

of the proof in H_2 of $(\neg\neg B \Rightarrow B)$

B_1 : $((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$

B_2 : $((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$

B_3 : $(\neg B \Rightarrow \neg B)$

B_4 : $((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$

B_5 : $(\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$

B_6 : $(\neg\neg B \Rightarrow B)$

Exercise 1

Exercise 1

Complete the proof presented in Example 1 by providing comments how each step of the proof was obtained.

ATTENTION

The solution presented on the next slide shows you how you will have to write details of your solutions on the **TESTS**

Solutions of other problems presented later are less detailed

Use them as exercises to write a detailed, complete solutions

Exercise 1 Solution

Solution

The comments that complete the proof are as follows.

B_1 : $((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$
Axiom $A3$ for $A = \neg B, B = B$

B_2 : $((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$
B_1 and **Lemma 1 (b)** for
$A = (\neg B \Rightarrow \neg\neg B)$, $B = (\neg B \Rightarrow \neg B), C = B$, i.e. we have
$((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B)) \vdash ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$

Exercise 1 Solution

B_3 : $(\neg B \Rightarrow \neg B)$

We proved for H_1 and hence for H_2 that $\vdash (A \Rightarrow A)$ and we substitute $A = \neg B$

B_4 : $((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$

B_2, B_3 and MP

B_5 : $(\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$

Axiom $A1$ for $A = \neg\neg B, B = \neg B$

B_6 : $(\neg\neg B \Rightarrow B)$

B_4, B_5 and **Lemma 1 (a)** for

$A = \neg\neg B, B = (\neg B \Rightarrow \neg\neg B), C = B$; i.e.

$(\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B)), ((\neg B \Rightarrow \neg\neg B) \Rightarrow B) \vdash (\neg\neg B \Rightarrow B)$

Proofs from Axioms Only

General remark

Observe that in steps B_2, B_3, B_5, B_6 we **call on** previously proved facts and use them as a part of our proof.

We can **obtain** a proof that uses only axioms by inserting previously constructed formal proofs of these facts into the places occupying by the steps B_2, B_3, B_5, B_6

For example in previously constructed proof of $(A \Rightarrow A)$ we **replace** A by $\neg B$ and **insert** such constructed proof of $(\neg B \Rightarrow \neg B)$ after step B_2

The last step of the inserted proof becomes now "old" step B_3 and we re-numerate all other steps accordingly

Proofs from Axioms Only

Here are consecutive first THREE steps of the proof of $(\neg\neg B \Rightarrow B)$

B_1 : $((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$

B_2 : $((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$

B_3 : $(\neg B \Rightarrow \neg B)$

We insert now the proof of $(\neg B \Rightarrow \neg B)$ after step B_2 and **erase** the B_3

The **last step** of the inserted proof becomes the **erased** B_3

Proofs from Axioms Only

A part of new **transformed** proof is

B_1 : $((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$ (Old B_1)

B_2 : $((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$ (Old B_2)

We insert here the proof from axioms only of Old B_3

B_3 : $((\neg B \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow \neg B)) \Rightarrow ((\neg B \Rightarrow (\neg B \Rightarrow \neg B)) \Rightarrow (\neg B \Rightarrow \neg B)))$, (New B_3)

B_4 : $(\neg B \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow \neg B))$

B_5: $((\neg B \Rightarrow (\neg B \Rightarrow \neg B)) \Rightarrow (\neg B \Rightarrow \neg B)))$

B_6: $(\neg B \Rightarrow (\neg B \Rightarrow \neg B))$

B_7: $(\neg B \Rightarrow \neg B)$ (Old B_3)

Proofs from Axioms Only

We repeat our procedure by **replacing** the step B_2 by its formal proof as defined in the proof of the **Lemma 1 (b)**

We continue the process for all other steps which involved application of the **Lemma 1** until we get a full **formal proof** from the axioms of H_2 only

Usually we don't do it and we don't need to do it, but it is important to remember that **it always can be done**

Example 2

Example 2

Here are consecutive steps

$$B_1, \; B_2, \; \ldots.., \; B_5$$

in a proof of $(B \Rightarrow \neg\neg B)$

B_1 $((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$

B_2 $(\neg\neg\neg B \Rightarrow \neg B)$

B_3 $((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$

B_4 $(B \Rightarrow (\neg\neg\neg B \Rightarrow B))$

B_5 $(B \Rightarrow \neg\neg B)$

Exercise 2

Complete the proof presented in **Example 2** by providing detailed comments how each step of the proof was obtained.

Solution

The comments that complete the proof are as follows.

B_1 $((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$
Axiom $A3$ for $A = B, B = \neg\neg B$

B_2 $(\neg\neg\neg B \Rightarrow \neg B)$
Example 1 for $B = \neg B$

Exercise 2

B_3 $((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$

B_1, B_2 and MP, i.e.

$$\frac{(\neg\neg\neg B \Rightarrow \neg B) ; ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))}{((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)}$$

B_4 $(B \Rightarrow (\neg\neg\neg B \Rightarrow B))$

Axiom $A1$ for $A = B, \quad B = \neg\neg\neg B$

B_5 $(B \Rightarrow \neg\neg B)$

B_3, B_4 and lemma 1**a** for $A = B, B = (\neg\neg\neg B \Rightarrow B), C = \neg\neg B$, i.e.

$$(B \Rightarrow (\neg\neg\neg B \Rightarrow B)), ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B) \vdash (B \Rightarrow \neg\neg B)$$

Example 3

Here are consecutive steps

$B_1,\ B_2,\ ...,\ B_{12}$ in a proof of $(\neg A \Rightarrow (A \Rightarrow B))$

B_1 $\neg A$

B_2 A

B_3 $(A \Rightarrow (\neg B \Rightarrow A))$

B_4 $(\neg A \Rightarrow (\neg B \Rightarrow \neg A))$

B_5 $(\neg B \Rightarrow A)$

B_6 $(\neg B \Rightarrow \neg A)$

B_7 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$

Example 3

B_8 $((\neg B \Rightarrow A) \Rightarrow B)$

B_9 B

B_{10} $\neg A, A \vdash B$

B_{11} $\neg A \vdash (A \Rightarrow B)$

B_{12} $(\neg A \Rightarrow (A \Rightarrow B))$

Exercise 3

1. Complete the proof from the Example 3 by providing comments how each step of the proof was obtained.

2. Prove that

$$\neg A, A \vdash B$$

Exercise 4

Example 4

Here are consecutive steps $B_1, ..., B_7$

in a proof of $((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$

B_1 $(\neg B \Rightarrow \neg A)$

B_2 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$

B_3 $(A \Rightarrow (\neg B \Rightarrow A))$

B_4 $((\neg B \Rightarrow A) \Rightarrow B)$

B_5 $(A \Rightarrow B)$

B_6 $(\neg B \Rightarrow \neg A) \vdash (A \Rightarrow B)$

B_7 $((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$

Exercise 4

Complete the proof from Example 4 by providing comments how each step of the proof was obtained

Example 5

Example 5

Here are consecutive steps $B_1, ..., B_9$

in a proof of $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

B_1 $(A \Rightarrow B)$

B_2 $(\neg\neg A \Rightarrow A)$

B_3 $(\neg\neg A \Rightarrow B)$

B_4 $(B \Rightarrow \neg\neg B)$

B_5 $(\neg\neg A \Rightarrow \neg\neg B)$

B_6 $((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$

B_7 $(\neg B \Rightarrow \neg A)$

B_8 $(A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$

B_9 $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

Exercise 5

Complete the proof of **Example 5** by providing comments how each step of the proof was obtained.

Solution

B_1 $(A \Rightarrow B)$
Hypothesis

B_2 $(\neg\neg A \Rightarrow A)$
Example 1 for $B = A$

B_3 $(\neg\neg A \Rightarrow B)$
Lemma 1 **a** for $A = \neg\neg A, B = A, C = B$

B_4 $(B \Rightarrow \neg\neg B)$
Example 2

Exercise 5

B_5 $(\neg\neg A \Rightarrow \neg\neg B)$
Lemma 1 **a** for $A = \neg\neg A, B = B, C = \neg\neg B$

B_6 $((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$
Example 4 for $B = \neg A, A = \neg B$

B_7 $(\neg B \Rightarrow \neg A)$
B_5, B_6 and MP

B_8 $(A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$
$B_1 - B_7$

B_9 $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$
Deduction Theorem

Example 6

Prove that $\vdash (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B))))$

Solution Here are consecutive steps of building the formal proof.

B_1 $A, (A \Rightarrow B) \vdash B$
by MP

B_2 $A \vdash ((A \Rightarrow B) \Rightarrow B)$
Deduction Theorem

B_3 $\vdash (A \Rightarrow ((A \Rightarrow B) \Rightarrow B))$
Deduction Theorem

B_4 $\vdash (((A \Rightarrow B) \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))$
Example 5 for $A = (A \Rightarrow B), B = B$

B_5 $\vdash (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B)))$
B_3 and B_4 and lemma 2**a** for
$A = A, B = ((A \Rightarrow B) \Rightarrow B), C = (\neg B \Rightarrow (\neg(A \Rightarrow B))$

Example 7

Here are consecutive steps $B_1, ..., B_{12}$

in a proof of $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$

B_1 $(A \Rightarrow B)$

B_2 $(\neg A \Rightarrow B)$

B_3 $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

B_4 $(\neg B \Rightarrow \neg A)$

B_5 $((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg\neg A))$

B_6 $(\neg B \Rightarrow \neg\neg A)$

B_7 $((\neg B \Rightarrow \neg\neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B)))$

Example 7

B_8 $((\neg B \Rightarrow \neg A) \Rightarrow B)$

B_9 B

B_{10} $(A \Rightarrow B), (\neg A \Rightarrow B) \vdash B$

B_{11} $(A \Rightarrow B) \vdash ((\neg A \Rightarrow B) \Rightarrow B)$

B_{12} $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$

Exercise 7

Complete the proof in Example 7 by providing comments how each step of the proof was obtained.

Exercise 7

Solution

B_1 $(A \Rightarrow B)$
Hypothesis

B_2 $(\neg A \Rightarrow B)$
Hypothesis

B_3 $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$
Example 5

B_4 $(\neg B \Rightarrow \neg A)$
B_1, B_3 and MP

B_5 $((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg\neg A))$
Example 5 for $A = \neg A, B = B$

B_6 $(\neg B \Rightarrow \neg\neg A)$
$B2, B_5$ and MP

Exercise 7

B_7 $((\neg B \Rightarrow \neg\neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B)))$
Axiom A3 for $B = B, A = \neg A$

B_8 $((\neg B \Rightarrow \neg A) \Rightarrow B)$
B_6, B_7 and MP

B_9 B
B_4, B_8 and MP

B_{10} $(A \Rightarrow B), (\neg A \Rightarrow B) \vdash B$
$B1 - B9$

B_{11} $(A \Rightarrow B) \vdash ((\neg A \Rightarrow B) \Rightarrow B)$
Deduction Theorem

B_{12} $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$
Deduction Theorem

Example 8

Here are consecutive steps $B_1, ..., B_3$

in a proof of $((\neg A \Rightarrow A) \Rightarrow A)$

B_1 $((\neg A \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow A) \Rightarrow A)))$

B_1 $(\neg A \Rightarrow \neg A)$

B_1 $((\neg A \Rightarrow A) \Rightarrow A))$

Exercise 8

Complete the proof of example 8 by providing comments how each step of the proof was obtained.

Solution

B_1 $((\neg A \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow A) \Rightarrow A)))$

Axiom A3 for $B = A$

B_1 $(\neg A \Rightarrow \neg A)$

Already proved $(A \Rightarrow A)$ for $A = \neg A$

B_1 $((\neg A \Rightarrow A) \Rightarrow A))$

B_1, B_2 and MP

LEMMA

We summarize all the formal proofs in H_2 provided in our Examples and Exercises in a form of a following Lemma

Lemma

The following formulas a **are provable** in H_2

1. $(A \Rightarrow A)$
2. $(\neg\neg B \Rightarrow B)$
3. $(B \Rightarrow \neg\neg B)$
4. $(\neg A \Rightarrow (A \Rightarrow B))$
5. $((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$
6. $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$
7. $(A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B)))$
8. $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$
9. $((\neg A \Rightarrow A) \Rightarrow A)$

Proof of Completeness Theorem

Formulas 1, 3, 4, and **7-9** from the set of provable formulas from the **Lemma** are all formulas we need together with H_2 axioms to **execute two proofs** of the **Completeness Theorem** for H_2

We present these proofs in Lecture 5a and Lecture 5b
The two proofs represent two different methods of proving the **Completeness Theorem**