CSE581
Computer Science Fundamentals: Theory

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Chapter 5
HILBERT PROOF SYSTEMS: Completeness of Classical Propositional Logic

PART 1: Hilbert Proof System $H_1$ and examples of applications of Deduction Theorem

PART 2: Proof of Deduction Theorem for System $H_1$

PART 3: System $H_2$ and examples of formal proofs in $H_2$
Hilbert proof systems are based on a language with implication and contain Modus Ponens as a rule of inference.

Modus Ponens is probably the oldest of all known rules of inference as it was already known to the Stoics in 3rd century B.C. and is also considered as the most natural to our intuitive thinking.

The proof systems containing Modus Ponens as the inference rule play a special role in logic.
Hilbert Proof Systems

Hilbert systems put major emphasis on logical axioms and keep the number of rules of inference at the minimum. Hilbert systems often admit the Modus Ponens as the sole rule of inference.

There are many proof systems that describe classical propositional logic, i.e. that are complete with respect to the classical semantics.

We present a Hilbert proof system for the classical propositional logic and discuss two ways of proving the Completeness Theorem for it.
Hilbert Proof Systems

The first proof is based on the one included in Elliott Mendelson’s book Introduction to Mathematical Logic. It is a constructive proof that shows how one can use the assumption that a formula $A$ is a tautology in order to construct its formal proof.

The second proof is non-constructive. Its importance lies in the fact that the methods it uses can be applied to the proof of completeness for classical predicate logic (chapter 9).

It also generalizes to some non-classical logics.
Hilbert Proof Systems

We prove completeness part of the Completeness Theorem by proving the converse implication to it.

We show how one can deduce that a formula $A$ is not a tautology from the fact that it does not have a proof.

It is hence called a counter-model construction proof.

Both proofs relay on the Deduction Theorem and so this is the first theorem we are now going to prove.
Hilbert Proof System $H_1$

We consider now a Hilbert proof system $H_1$ based on a this is language with implication as the only connective, with two logical axioms, and with Modus Ponens as a sole rule of inference.
Hilbert Proof System $H_1$

We define Hilbert system $H_1$ as follows

$$H_1 = (L_{\Rightarrow}, \mathcal{F}, \{A_1, A_2\}, MP)$$

**A1** (Law of simplification)

$$(A \Rightarrow (B \Rightarrow A))$$

**A2** (Frege’s Law)

$$(((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))))$$

**MP** is the Modus Ponens rule

$$(MP) \quad A ; (A \Rightarrow B) \quad B$$

where $A, B, C$ are any formulas from $\mathcal{F}$
Formal Proofs in $H_1$

Finding **formal proofs** in this system requires some ingenuity. **The formal proof** of $(A \Rightarrow A)$ in $H_1$ is a sequence $B_1, B_2, B_3, B_4, B_5$ as defined below.

- $B_1 : \left( (A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)) \right)$,
  - axiom A2 for $A = A, B = (A \Rightarrow A)$, and $C = A$

- $B_2 : (A \Rightarrow ((A \Rightarrow A) \Rightarrow A))$,
  - axiom A1 for $A = A, B = (A \Rightarrow A)$

- $B_3 : \left( (A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A) \right)$,
  - MP application to $B_1$ and $B_2$

- $B_4 : (A \Rightarrow (A \Rightarrow A))$,
  - axiom A1 for $A = A, B = A$

- $B_5 : (A \Rightarrow A)$
  - MP application to $B_3$ and $B_4$
A general procedure for automated search for proofs in a proof system $S$ can be stated as follows:

Let $B$ be an expression of the system $S$ that is not an axiom.

If $B$ has a proof in $S$, $B$ must be the conclusion of one of the inference rules.

Let’s say it is a rule $r$.

We find all its premisses, i.e., we evaluate $r^{-1}(B)$.

If all premisses are axioms, the proof is found.

Otherwise, we repeat the procedure for any premiss that is not an axiom.
Search for Proof by the Means of MP

The **MP** rule says:

given two formulas $A$ and $(A \Rightarrow B)$ we conclude a formula $B$

**Assume** now that and want to find a **proof** of a formula $B$

If $B$ is an **axiom**, we have the **proof**; the formula itself

If $B$ **is not** an axiom, it had to be obtained by the application of the Modus Ponens rule to certain two formulas $A$ and $(A \Rightarrow B)$ and there is infinitely many of such formulas!

The proof system $H_1$ **is not** syntactically decidable
Semantic Links

Semantic Link 1

System $H_1$ is sound under classical semantics and $H_1$ is not sound under $K$ semantics.

Soundness Theorem for $H_1$

For any $A \in \mathcal{F}$, if $\vdash_{H_1} A$, then $\models A$
Semantic Links

Semantic Link 2

The system $H_1$ is not complete under classical semantics.

Not all classical tautologies have a proof in $H_1$.

We proved that can’t define negation in term of implication alone and so for example, a basic tautology $(\neg\neg A \Rightarrow A)$ is not provable in $H_1$, i.e.

$$\not\exists_{H_1} (\neg\neg A \Rightarrow A)$$
Proof from Hypothesis

Given a proof system $S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$
While proving expressions we often use some extra information available, besides the axioms of the proof system
This extra information is called hypothesis in the proof

Let $\Gamma \subseteq \mathcal{E}$ be any set expressions called hypothesis

We write $\Gamma \vdash_S E$ to denote that
"$E$ has a proof in $S$ from the set $\Gamma$ and the logical axioms $LA$"
Formal Definition

Definition
We say that $E \in \mathcal{E}$ has a \textbf{formal proof} in $S$ from the set $\Gamma$ and the logical axioms $\text{LA}$ and denote it as $\Gamma \vdash_S E$ if and only if there is a sequence $A_1, \ldots, A_n$ of expressions from $\mathcal{E}$, such that

$$A_1 \in \text{LA} \cup \Gamma, \quad A_n = E$$

and for each $1 < i \leq n$, either $A_i \in \text{LA} \cup \Gamma$ or $A_i$ is a \textbf{direct consequence} of some of the \textbf{preceding} expressions by virtue of \textbf{one of the rules} of inference of $S$.
Special Cases

Case 1: \( \Gamma \subseteq \mathcal{E} \) is a finite set and \( \Gamma = \{B_1, B_2, ..., B_n\} \)

We write \( B_1, B_2, ..., B_n \vdash_S E \)

instead of \( \{B_1, B_2, ..., B_n\} \vdash_S E \)

Case 2: \( \Gamma = \emptyset \)

By the definition of a proof of \( E \) from \( \Gamma \), \( \emptyset \vdash_S E \) means that in the proof of \( E \) we use only the logical axioms \( \text{LA} \) of \( S \)

We hence write \( \vdash_S E \)

to denote that \( E \) has a proof from \( \Gamma = \emptyset \)
Proof from Hypothesis in $H_1$

Show that

$$(A \Rightarrow B), \ (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)$$

We construct a formal proof

$$B_1, B_2, \ldots, B_7$$

of $(A \Rightarrow C)$ from hypothesis $(A \Rightarrow B)$ and $(B \Rightarrow C)$ as follows
Proof from Hypothesis in $H_1$

$B_1 : (B \Rightarrow C), \quad B_2 : (A \Rightarrow B),$

hypothesis hypothesis

$B_3 : ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))),$
axiom A2

$B_4 : ((B \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))),$
axiom A1 for $A = (B \Rightarrow C), \; B = A$

$B_5 : (A \Rightarrow (B \Rightarrow C)),$
$B_1$ and $B_4$ and MP

$B_6 : ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)), \quad B_7 : (A \Rightarrow C)$

MP
Deduction Theorem

In mathematical arguments, one often proves a statement $B$ on the assumption of some other statement $A$ and then concludes that we have proved the implication "if $A$, then $B". This reasoning is justified by a following theorem, called a Deduction Theorem.

Reminder
We write $\Gamma, A \vdash B$ for $\Gamma \cup \{A\} \vdash B$
In general, we write $\Gamma, A_1, A_2, ..., A_n \vdash B$
for $\Gamma \cup \{A_1, A_2, ..., A_n\} \vdash B$
Deduction Theorem for $H_1$

**Deduction Theorem** for $H_1$

For any $A, B \in \mathcal{F}$ and $\Gamma \subseteq \mathcal{F}$

$$\Gamma, A \vdash_{H_1} B \quad \text{if and only if} \quad \Gamma \vdash_{H_1} (A \Rightarrow B)$$

In particular

$$A \vdash_{H_1} B \quad \text{if and only if} \quad \vdash_{H_1} (A \Rightarrow B)$$
Formal Proofs

The proof of the following **Lemma** provides a good example of multiple applications of the **Deduction Theorem**

**Lemma**
For any \( A, B, C \in \mathcal{F} \),

(a) \( (A \rightarrow B), (B \rightarrow C) \vdash_{H_1} (A \rightarrow C) \),

(b) \( (A \rightarrow (B \rightarrow C)) \vdash_{H_1} (B \rightarrow (A \rightarrow C)) \)

Observe that by **Deduction Theorem** we can re-write (a) as

(a') \( (A \rightarrow B), (B \rightarrow C), A \vdash_{H_1} C \)
**Proof of (a')**

We construct a formal proof

\[ B_1, B_2, B_3, B_4, B_5 \]

of \((A \Rightarrow B), (B \Rightarrow C), A \vdash_{H_1} C\) as follows.

- **\(B_1\)**: \((A \Rightarrow B)\)
  - hypothesis
- **\(B_2\)**: \((B \Rightarrow C)\)
  - hypothesis
- **\(B_3\)**: \(A\)
  - hypothesis
- **\(B_4\)**: \(B\)
- **\(B_5\)**: \(C\)

\(B_1, B_3\) and MP

\(B_2, B_4\) and MP
Thus we proved by **Deduction Theorem** that (a) holds, i.e.

\[(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)\]

**Proof** of Lemma part (b)

By **Deduction Theorem** we have that

\[(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))\]

if and only if

\[(A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C)\]
Formal Proofs

We construct a formal proof

\[ B_1, B_2, B_3, B_4, B_5, B_6, B_7 \]

of \( (A \Rightarrow (B \Rightarrow C)) \), \( B \vdash_{H_1} (A \Rightarrow C) \) as follows.

\[ B_1 : \quad (A \Rightarrow (B \Rightarrow C)) \]

hypothesis

\[ B_2 : \quad B \]

hypothesis

\[ B_3 : \quad ((B \Rightarrow (A \Rightarrow B)) \]

A1 for \( A = B, B = A \)

\[ B_4 : \quad (A \Rightarrow B) \]

\( B_2, B_3 \) and MP
Formal Proofs

\[ B_5 : ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))) \]

axiom A2

\[ B_6 : ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)) \]

\[ B_1, B_5 \text{ and } MP \]

\[ B_7 : (A \Rightarrow C) \]

Thus we proved by Deduction Theorem that

\[ (A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C)) \]
Simpler Proof

Here is a simpler proof of Lemma part (b)

We apply the Deduction Theorem twice, i.e. we get

\[(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))\]

if and only if

\[(A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C)\]

if and only if

\[(A \Rightarrow (B \Rightarrow C)), B, A \vdash_{H_1} C\]
Simpler Proof

We now construct a proof of \((A \implies (B \implies C)), B, A \vdash_{H_1} C\) as follows

\(B_1: (A \implies (B \implies C))\)
- hypothesis

\(B_2: B\)
- hypothesis

\(B_3: A\)
- hypothesis

\(B_4: (B \implies C)\)
\(B_1, B_3\) and (MP)

\(B_5: C\)
\(B_2, B_4\) and (MP)
CONSEQUENCE OPERATION
Review
Definition: Consequences of $\Gamma$

Given a proof system

$$S = (\mathcal{L}, \mathcal{E}, LA, R)$$

For any $\Gamma \subseteq \mathcal{E}$, and $A \in \mathcal{E}$,

If $\Gamma \vdash_S A$, then $A$ is called a consequence of $\Gamma$ in $S$.

We denote by $\text{Cn}_S(\Gamma)$ the set of all consequences of $\Gamma$ in $S$, i.e. we put

$$\text{Cn}_S(\Gamma) = \{A \in \mathcal{E} : \Gamma \vdash_S A\}$$
Definition: Consequence Operation

Observe that by defining a consequence of $\Gamma$ in $S$, we define in fact a function which to every set $\Gamma \subseteq \mathcal{E}$ assigns a set of all its consequences $Cn_S(\Gamma)$.

We denote this function by $Cn_S$ and adopt the following Definition:

Any function

$$Cn_S : 2^\mathcal{E} \rightarrow 2^\mathcal{E}$$

such that for every $\Gamma \in 2^\mathcal{E}$

$$Cn_S(\Gamma) = \{ A \in \mathcal{E} : \Gamma \vdash_S A \}$$

is called the consequence operation determined by $S$. 
Consequence Operation: Monotonicity

Take any consequence operation determined by $S$

$$Cn_S : 2^\mathcal{E} \rightarrow 2^\mathcal{E}$$

Monotonicity Property
For any sets $\Gamma, \Delta$ of expressions of $S$,
if $\Gamma \subseteq \Delta$ then $Cn_S(\Gamma) \subseteq Cn_S(\Delta)$

Exercise: write the proof;
it follows directly from the definition of $Cn_S$ and definition of the formal proof
Consequence Operation: Transitivity

Take any consequence operation

\[ Cn_S : 2^E \rightarrow 2^E \]

Transitivity Property
For any sets \( \Gamma_1, \Gamma_2, \Gamma_3 \) of expressions of \( S \),
if \( \Gamma_1 \subseteq Cn_S(\Gamma_2) \) and \( \Gamma_2 \subseteq Cn_S(\Gamma_3) \), then \( \Gamma_1 \subseteq Cn_S(\Gamma_3) \)

Exercise: write the proof;
it follows directly from the definition of \( Cn_S \) and definition of the formal proof
Consequence Operation: Finiteness

Take any consequence operation determined by

\[ \text{Cn}_S : 2^\mathcal{E} \rightarrow 2^\mathcal{E} \]

Finiteness Property

For any expression \( A \in \mathcal{E} \) and any set \( \Gamma \subseteq \mathcal{E} \),\n\[ A \in \text{Cn}_S(\Gamma) \text{ if and only if } \text{there is a finite subset } \Gamma_0 \text{ of } \Gamma \text{ such that } A \in \text{Cn}_S(\Gamma_0) \]

Exercise: write the proof;
it follows directly from the definition of \( \text{Cn}_S \) and definition of the formal proof
Proof Deduction Theorem for $H_1$
The Deduction Theorem

As we now fix the proof system to be $H_1$, we write $A \vdash B$ instead of $A \vdash_{H_1} B$

**Deduction Theorem** (Herbrand, 1930) for $H_1$

For any formulas $A, B \in \mathcal{F}$,

If $A \vdash B$, then $\vdash (A \Rightarrow B)$

**Deduction Theorem** (General case) for $H_1$

For any formulas $A, B \in \mathcal{F}$, $\Gamma \subseteq \mathcal{F}$$

$\Gamma, A \vdash B$ if and only if $\Gamma \vdash (A \Rightarrow B)$

**Proof:**

**Part 1** We first prove the ”if” part:

If $\Gamma, A \vdash B$ then $\Gamma \vdash (A \Rightarrow B)$
Proof of The Deduction Theorem

Assume that

$$\Gamma, A \vdash B$$

i.e. that we have a formal proof

$$B_1, B_2, ..., B_n$$

of $B$ from the set of formulas $\Gamma \cup \{A\}$

We have to show that

$$\Gamma \vdash (A \Rightarrow B)$$
Proof of The Deduction Theorem

In order to prove that \( \Gamma \vdash (A \Rightarrow B) \) follows from \( \Gamma, A \vdash B \),
we prove a stronger statement, namely that

\[ \Gamma \vdash (A \Rightarrow B_i) \]

for any \( B_i, \ 1 \leq i \leq n \) in the formal proof \( B_1, B_2, ..., B_n \) of \( B \)
also follows from \( \Gamma, A \vdash B \).

Hence in particular case, when \( i = n \) we will obtain that
\( \Gamma \vdash (A \Rightarrow B) \) follows from \( \Gamma, A \vdash B \)
and that will end the proof of Part 1.
Base Step

The proof of Part 1 is conducted by mathematical induction on $i$, for $1 \leq i \leq n$

Step 1 $i = 1$ (base step)

Observe that when $i = 1$, it means that the formal proof $B_1, B_2, ..., B_n$ contains only one element $B_1$

By the definition of the formal proof from $\Gamma \cup \{A\}$, we have that

1. $B_1$ is a logical axiom, or $B_1 \in \Gamma$, or
2. $B_1 = A$

This means that $B_1 \in \{A1, A2\} \cup \Gamma \cup \{A\}$
Base Step

Now we have two cases to consider.

Case1: $B_1 \in \{A_1, A_2\} \cup \Gamma$

Observe that $(B_1 \Rightarrow (A \Rightarrow B_1))$ is the axiom $A_1$

By assumption $B_1 \in \{A_1, A_2\} \cup \Gamma$

We get the required proof of $(A \Rightarrow B_1)$ from $\Gamma$

by the following application of the Modus Ponens rule

\[(\text{MP}) \quad \frac{B_1; (B_1 \Rightarrow (A \Rightarrow B_1))}{(A \Rightarrow B_1)}\]
Case 2: \( B_1 = A \)
When \( B_1 = A \) then to prove \( \Gamma \vdash (A \Rightarrow B_1) \)
This means we have to prove
\[
\Gamma \vdash (A \Rightarrow A)
\]
This holds by monotoncity of the consequence and the fact that we have shown that
\[
\vdash (A \Rightarrow A)
\]
The above cases conclude the proof for \( i = 1 \) of
\[
\Gamma \vdash (A \Rightarrow B_i)
\]
Inductive Step

Assume that

\[ \Gamma \vdash (A \Rightarrow B_k) \]

for all \( k < i \) (strong induction)

We will show that using this fact we can conclude that also

\[ \Gamma \vdash (A \Rightarrow B_i) \]
Inductive Step

Consider a formula $B_i$ in the formal proof $B_1, B_2, ..., B_n$

By definition of the formal proof we have to show the following tow cases

Case 1: $B_i \in \{A_1, A_2\} \cup \Gamma \cup \{A\}$ and

Case 2: $B_i$ follows by MP from certain $B_j, B_m$ such that $j < m < i$

Consider now the Case 1: $B_i \in \{A_1, A_2\} \cup \Gamma \cup \{A\}$

The proof of $(A \Rightarrow B_i)$ from $\Gamma$ in this case is obtained from the proof of the Step $i = 1$ by replacement $B_1$ by $B_i$

and is omitted here as a straightforward repetition
Inductive Step

Case 2:

\(B_i\) is a **conclusion** of (MP)

If \(B_i\) is a conclusion of (MP), then we must have two formulas \(B_j, B_m\) in the formal proof

\[ B_1, B_2, ..., B_n \]

such that \(j < i, m < i, j \neq m\) and

\[
\text{(MP)} \quad \frac{B_j ; B_m}{B_i}
\]
Inductive Step

By the **inductive assumption** the formulas $B_j, B_m$ are such that $\Gamma \vdash (A \Rightarrow B_j)$ and $\Gamma \vdash (A \Rightarrow B_m)$

Moreover, by the definition of (MP) rule, the formula $B_m$ has to have a form $(B_j \Rightarrow B_i)$
This means that

$$B_m = (B_j \Rightarrow B_i)$$

The **inductive assumption** can be re-written as follows

$$\Gamma \vdash (A \Rightarrow (B_j \Rightarrow B_i))$$

for $j < i$
Inductive Step

**Observe** now that the formula

$$(((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))))$$

is a *substitution of the axiom A2* and hence **has a proof** in our system.

By the monotonicity of the consequence, it also has a proof from the set $\Gamma$, i.e.

$$\Gamma \vdash (((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))))$$
Inductive Step

We know that

\[ \Gamma \vdash ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))) \]

Applying the rule MP i.e. performing the following

\[
(A \Rightarrow (B_j \Rightarrow B_i)) ; ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))) \\
\hline 
((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))
\]

we get that also

\[ \Gamma \vdash (A \Rightarrow B_j \Rightarrow (A \Rightarrow B_i)) \]
Inductive Step

Applying again the rule MP i.e. performing the following

\[(A \Rightarrow B_j) ; ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))\]
\[\Rightarrow (A \Rightarrow B_i)\]

we get that

\[\Gamma \vdash (A \Rightarrow B_i)\]

what ends the proof of the inductive step
Proof of the Deduction Theorem

By the mathematical induction principle, we have proved that

$$\Gamma \vdash (A \Rightarrow B_i), \quad \text{for all } 1 \leq i \leq n$$

In particular it is true for $i = n$, i.e. for $B_n = B$

and we proved that

$$\Gamma \vdash (A \Rightarrow B)$$

This ends the proof of the first part of the Deduction Theorem:

If $\Gamma, A \vdash B$, then $\Gamma \vdash (A \Rightarrow B)$
Proof of the Deduction Theorem

The proof of the second part, i.e. of the inverse implication:

If $\Gamma \vdash (A \Rightarrow B)$, then $\Gamma, A \vdash B$

is straightforward and goes as follows.

Assume that $\Gamma \vdash (A \Rightarrow B)$

By the monotonicity of the consequence we have also that $\Gamma, A \vdash (A \Rightarrow B)$

Obviously $\Gamma, A \vdash A$

Applying Modus Ponens to the above, we get the proof of $B$ from $\{\Gamma, A\}$

We have hence proved that $\Gamma, A \vdash B$
Proof of the Deduction Theorem

This ends the proof of

**Deduction Theorem**  (General case ) for $H_1$

For any formulas $A, B \in \mathcal{F}$ and any $\Gamma \subseteq \mathcal{F}$

$$\Gamma, A \vdash B \quad \text{if and only if} \quad \Gamma \vdash (A \Rightarrow B)$$

The particular case we get also the particular case

**Deduction Theorem**  (Herbrand, 1930) for $H_1$

For any formulas $A, B \in \mathcal{F},$

If $A \vdash B,$  then $\vdash (A \Rightarrow B)$

is obtained from the above by assuming that the set $\Gamma$ is empty
Classical Propositional Proof System $H_2$
Hilbert System $H_2$

The proof system $H_1$ is **sound** and strong enough to prove the Deduction Theorem, but it is **not complete**.

We extend now its **language** and the set of **logical axioms** to a **complete set of axioms**.

**We define** a system $H_2$ that is **complete** with respect to the classical semantics.

The **proof of completeness theorem** is be presented in the next chapter.
Hilbert System $H_2$ Definition

Definition

$$H_2 = ( \mathcal{L}_{\{\Rightarrow, \neg\}}, \mathcal{F}, \{A_1, A_2, A_3\} (MP) )$$

A1  (Law of simplification)
$$(A \Rightarrow (B \Rightarrow A))$$

A2  (Frege’s Law)
$$( (A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)) )$$

A3  $$( (\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B) )$$

MP  (Rule of inference)

$$(MP) \quad \frac{A ; (A \Rightarrow B)}{B}$$

where $A, B, C$ are any formulas of the propositional language $\mathcal{L}_{\{\Rightarrow, \neg\}}$
Observation 1
The proof system $H_2$ is obtained by adding axiom $A_3$ to the system $H_1$

Observation 2
The language of $H_2$ is obtained by adding the connective $\neg$ to the language of $H_1$

Observation 3
The use of axioms $A_1, A_2$ in the proof of Deduction Theorem for the system $H_1$ is independent of the connective $\neg$ added to the language of $H_1$

Observation 4
Hence the proof of the Deduction Theorem for the system $H_1$ can be repeated as it is for the system $H_2$
Deduction Theorem for System $H_2$

Observations 1-4 prove that the Deduction Theorem holds for system $H_2$.

**Deduction Theorem for $H_2$**

For any $\Gamma \subseteq \mathcal{F}$ and $A, B \in \mathcal{F}$

$$\Gamma, A \vdash_{H_2} B \text{ if and only if } \Gamma \vdash_{H_2} (A \Rightarrow B)$$

In particular

$$A \vdash_{H_2} B \text{ if and only if } \vdash_{H_2} (A \Rightarrow B)$$
Soundness and Completeness Theorems

We get by easy verification

**Soundness Theorem** $H_2$

For every formula $A \in \mathcal{F}$

if $\vdash_{H_2} A$ then $\models A$

We prove in the next Lecture, that $H_2$ is also complete, i.e. we prove

**Completeness Theorem** for $H_2$

For every formula $A \in \mathcal{F}$,

$\vdash_{H_2} A$ if and only if $\models A$
The proof of completeness theorem (for a given semantics) is always a main point in creation of any new logic.

There are many techniques to prove it, depending on the proof system, and on the semantics we define for it.

We present in Lecture 5a and Lecture 5b two proofs of the Completeness Theorem for the system $H_2$.

These proofs use very different techniques, hence the reason of presenting both of them.
FORMAL PROOFS IN $H_2$
Examples and Exercises

We present now some examples of formal proofs in $H_2$

There are two reasons for presenting them.

**First reason** is that all formulas we prove here to be provable play a crucial role in the proof of Completeness Theorem for $H_2$

**The second reason** is that they provide a "training ground" for a reader to learn how to develop formal proofs

For this reason we write some proofs in a full detail and we leave some for the reader to complete in a way explained in the following example.
Important Lemma

We write $\vdash$ instead of $\vdash_{H_2}$ for the sake of simplicity

Reminder
In the construction of the formal proofs we often use the Deduction Theorem and the following Lemma 1 they was proved in previous section

Lemma 1

(a) $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C)$

(b) $(A \Rightarrow (B \Rightarrow C)) \vdash_{H_2} ((B \Rightarrow (A \Rightarrow C))$
Example 1

Here are consecutive steps

\[ B_1, \ldots, B_5, \ B_6 \]

of the proof in \( H_2 \) of \((\neg\neg B \Rightarrow B)\)

\[ B_1 : \ ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B)) \]

\[ B_2 : \ ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B)) \]

\[ B_3 : \ (\neg B \Rightarrow \neg B) \]

\[ B_4 : \ ((\neg B \Rightarrow \neg\neg B) \Rightarrow B) \]

\[ B_5 : \ (\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B)) \]

\[ B_6 : \ (\neg\neg B \Rightarrow B) \]
Exercise 1

Complete the proof presented in Example 1 by providing comments how each step of the proof was obtained.

ATTENTION

The solution presented on the next slide shows you how you will have to write details of your solutions on the TESTS Solutions of other problems presented later are less detailed. Use them as exercises to write a detailed, complete solutions.
Exercise 1 Solution

Solution

The comments that complete the proof are as follows.

\( B_1 : (\neg B \Rightarrow \neg \neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B) \)

Axiom A3 for \( A = \neg B, B = B \)

\( B_2 : (\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg \neg B) \Rightarrow B) \)

\( B_1 \) and Lemma 1 (b) for

\( A = (\neg B \Rightarrow \neg \neg B), \ B = (\neg B \Rightarrow \neg B), \ C = B, \) i.e. we have

\( ((\neg B \Rightarrow \neg \neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B)) \vdash ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg \neg B) \Rightarrow B)) \)
Exercise 1 Solution

\( B_3 : (\neg B \Rightarrow \neg B) \)
We proved for \( H_1 \) and hence for \( H_2 \) that \( \vdash (A \Rightarrow A) \) and we substitute \( A = \neg B \)

\( B_4 : ( (\neg B \Rightarrow \neg \neg B) \Rightarrow B ) \)
\( B_2, B_3 \) and MP

\( B_5 : (\neg \neg B \Rightarrow (\neg B \Rightarrow \neg \neg B)) \)
Axiom A1 for \( A = \neg \neg B, B = \neg B \)

\( B_6 : (\neg \neg B \Rightarrow B) \)
\( B_4, B_5 \) and Lemma 1 (a) for
\( A = \neg \neg B, B = (\neg B \Rightarrow \neg \neg B), C = B; \) i.e.
\( (\neg \neg B \Rightarrow (\neg B \Rightarrow \neg \neg B)), ((\neg B \Rightarrow \neg \neg B) \Rightarrow B) \vdash (\neg \neg B \Rightarrow B) \)
Proofs from Axioms Only

General remark

Observe that in steps $B_2, B_3, B_5, B_6$ we call on previously proved facts and use them as a part of our proof.

We can obtain a proof that uses only axioms by inserting previously constructed formal proofs of these facts into the places occupying by the steps $B_2, B_3, B_5, B_6$

For example in previously constructed proof of $(A \Rightarrow A)$ we replace $A$ by $\neg B$ and insert such constructed proof of $(\neg B \Rightarrow \neg B)$ after step $B_2$

The last step of the inserted proof becomes now ”old” step $B_3$ and we re-numerate all other steps accordingly
Proofs from Axioms Only

Here are consecutive first THREE steps of the proof of $(\neg\neg B \Rightarrow B)$

$B_1 : ( (\neg B \Rightarrow \neg \neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B) )$

$B_2 : ( (\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg \neg B) \Rightarrow B) )$

$B_3 : ( \neg B \Rightarrow \neg B )$

We insert now the proof of $(\neg B \Rightarrow \neg B)$ after step $B_2$ and erase the $B_3$

The last step of the inserted proof becomes the erased $B_3$
Proofs from Axioms Only

A part of new transformed proof is

\[ B_1 : \ ( ( \neg B \Rightarrow \neg \neg B ) \Rightarrow ( ( \neg B \Rightarrow \neg B ) \Rightarrow B ) ) \ \ ( \text{Old } B_1 ) \]

\[ B_2 : \ ( ( \neg B \Rightarrow \neg B ) \Rightarrow ( ( \neg B \Rightarrow \neg \neg B ) \Rightarrow B ) ) \ \ ( \text{Old } B_2 ) \]

We insert here the proof from axioms only of Old \( B_3 \)

\[ B_3 : \ ( ( \neg B \Rightarrow ( ( \neg B \Rightarrow \neg B ) \Rightarrow \neg B ) ) ) \Rightarrow ( ( \neg B \Rightarrow ( \neg B \Rightarrow \neg B ) \Rightarrow \neg B ) ) \Rightarrow ( \neg B \Rightarrow \neg B ) ) ) \)

\[ ( \text{New } B_3 ) \]

\[ B_4 : \ ( \neg B \Rightarrow ( ( \neg B \Rightarrow \neg B ) \Rightarrow \neg B ) ) \]

\[ B_5 : \ ( ( \neg B \Rightarrow ( \neg B \Rightarrow \neg B ) ) ) \Rightarrow ( \neg B \Rightarrow \neg B ) ) \)

\[ B_6 : \ ( \neg B \Rightarrow ( \neg B \Rightarrow \neg B ) ) \]

\[ B_7 : \ ( \neg B \Rightarrow \neg B ) \ \ ( \text{Old } B_3 ) \]
Proofs from Axioms Only

We repeat our procedure by replacing the step $B_2$ by its formal proof as defined in the proof of the Lemma 1 (b)

We continue the process for all other steps which involved application of the Lemma 1 until we get a full formal proof from the axioms of $H_2$ only

Usually we don’t do it and we don’t need to do it, but it is important to remember that it always can be done
Example 2

Here are consecutive steps

\[ B_1, B_2, \ldots, B_5 \]

in a proof of \( (B \Rightarrow \neg
\neg B) \)

\[ B_1 \quad ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)) \]

\[ B_2 \quad (\neg\neg\neg B \Rightarrow \neg B) \]

\[ B_3 \quad ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B) \]

\[ B_4 \quad (B \Rightarrow (\neg\neg\neg B \Rightarrow B)) \]

\[ B_5 \quad (B \Rightarrow \neg\neg B) \]
Exercise 2

Exercise 2

Complete the proof presented in Example 2 by providing detailed comments how each step of the proof was obtained.

Solution

The comments that complete the proof are as follows.

$B_1 \quad ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$

Axiom A3 for $A = B, B = \neg\neg B$

$B_2 \quad (\neg\neg\neg B \Rightarrow \neg B)$

Example 1 for $B = \neg B$
Exercise 2

\(B_3\) \(((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)\)

\(B_1, B_2\) and MP, i.e.

\(((\neg\neg\neg B \Rightarrow \neg B);((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))

\(((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)\)

\(B_4\) \((B \Rightarrow (\neg\neg\neg B \Rightarrow B))\)

Axiom A1 for \(A = B, \ B = \neg\neg\neg B\)

\(B_5\) \((B \Rightarrow \neg\neg B)\)

\(B_3, B_4\) and lemma 1a for \(A = B, B = (\neg\neg\neg B \Rightarrow B), C = \neg\neg B\), i.e.

\((B \Rightarrow (\neg\neg\neg B \Rightarrow B)), ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B) \vdash (B \Rightarrow \neg\neg B)\)
Example 3

Example 3

Here are consecutive steps

\[ B_1, B_2, ..., B_{12} \] in a proof of \((\lnot A \Rightarrow (A \Rightarrow B))\)

\begin{align*}
B_1 & \quad \lnot A \\
B_2 & \quad A \\
B_3 & \quad (A \Rightarrow (\lnot B \Rightarrow A)) \\
B_4 & \quad (\lnot A \Rightarrow (\lnot B \Rightarrow \lnot A)) \\
B_5 & \quad (\lnot B \Rightarrow A) \\
B_6 & \quad (\lnot B \Rightarrow \lnot A) \\
B_7 & \quad ((\lnot B \Rightarrow \lnot A) \Rightarrow ((\lnot B \Rightarrow A) \Rightarrow B))
\end{align*}
Example 3

$B_8 \quad ((\neg B \Rightarrow A) \Rightarrow B)$
$B_9 \quad B$
$B_{10} \quad \neg A, A \vdash B$
$B_{11} \quad \neg A \vdash (A \Rightarrow B)$
$B_{12} \quad (\neg A \Rightarrow (A \Rightarrow B))$

Exercise 3

1. Complete the proof from the Example 3 by providing comments how each step of the proof was obtained.

2. Prove that

    \[\neg A, A \vdash B\]
Exercise 4

Example 4

Here are consecutive steps $B_1, ..., B_7$ in a proof of $((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$

$B_1$ $(\neg B \Rightarrow \neg A)$

$B_2$ $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$

$B_3$ $(A \Rightarrow (\neg B \Rightarrow A))$

$B_4$ $((\neg B \Rightarrow A) \Rightarrow B)$

$B_5$ $(A \Rightarrow B)$

$B_6$ $(\neg B \Rightarrow \neg A) \vdash (A \Rightarrow B)$

$B_7$ $((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$

Exercise 4

Complete the proof from Example 4 by providing comments how each step of the proof was obtained
Example 5

Here are consecutive steps \( B_1, \ldots, B_9 \) in a proof of \( ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)) \)

- **B_1** \( (A \Rightarrow B) \)
- **B_2** \( (\neg \neg A \Rightarrow A) \)
- **B_3** \( (\neg \neg A \Rightarrow B) \)
- **B_4** \( (B \Rightarrow \neg \neg B) \)
- **B_5** \( (\neg \neg A \Rightarrow \neg \neg B) \)
- **B_6** \( ((\neg \neg A \Rightarrow \neg \neg B) \Rightarrow (\neg B \Rightarrow \neg A)) \)
- **B_7** \( (\neg B \Rightarrow \neg A) \)
- **B_8** \( (A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A) \)
- **B_9** \( ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)) \)
Exercise 5

Exercise 5
Complete the proof of Example 5 by providing comments how each step of the proof was obtained.

Solution

\( B_1 \quad (A \Rightarrow B) \)
Hypothesis

\( B_2 \quad (\neg\neg A \Rightarrow A) \)
Example 1 for \( B = A \)

\( B_3 \quad (\neg\neg A \Rightarrow B) \)
Lemma 1 a for \( A = \neg\neg A, B = A, C = B \)

\( B_4 \quad (B \Rightarrow \neg\neg B) \)
Example 2
Exercise 5

\[ B_5 \quad (\neg\neg A \Rightarrow \neg\neg B) \]

Lemma 1 a for \( A = \neg\neg A, B = B, C = \neg\neg B \)

\[ B_6 \quad ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A)) \]

Example 4 for \( B = \neg A, A = \neg B \)

\[ B_7 \quad (\neg B \Rightarrow \neg A) \]

\( B_5, B_6 \) and MP

\[ B_8 \quad (A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A) \]

\( B_1 \rightarrow B_7 \)

\[ B_9 \quad ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)) \]

Deduction Theorem
Example 6

Example 6

Prove that \( \vdash (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B)))) \)

Solution

Here are consecutive steps of building the formal proof.

\[ B_1 \quad A, (A \Rightarrow B) \vdash B \]

by MP

\[ B_2 \quad A \vdash ((A \Rightarrow B) \Rightarrow B) \]

Deduction Theorem

\[ B_3 \quad \vdash (A \Rightarrow ((A \Rightarrow B) \Rightarrow B)) \]

Deduction Theorem

\[ B_4 \quad \vdash (((A \Rightarrow B) \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B))) \]

Example 5 for \( A = (A \Rightarrow B), B = B \)

\[ B_5 \quad \vdash (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B)))) \]

\( B_3 \) and \( B_4 \) and lemma 2a for

\( A = A, B = ((A \Rightarrow B) \Rightarrow B), C = (\neg B \Rightarrow (\neg(A \Rightarrow B))) \)
Here are consecutive steps $B_1, \ldots, B_{12}$ in a proof of $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$.

- $B_1$ (\(A \Rightarrow B\))
- $B_2$ (\(\neg A \Rightarrow B\))
- $B_3$ (\(((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))\))
- $B_4$ (\(\neg B \Rightarrow \neg A\))
- $B_5$ (\(((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg\neg A))\))
- $B_6$ (\(\neg B \Rightarrow \neg\neg A\))
- $B_7$ (\(((\neg B \Rightarrow \neg\neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B))\))
Example 7

$B_8$  $((\neg B \Rightarrow \neg A) \Rightarrow B)$

$B_9$  $B$

$B_{10}$  $(A \Rightarrow B), (\neg A \Rightarrow B) \vdash B$

$B_{11}$  $(A \Rightarrow B) \vdash ((\neg A \Rightarrow B) \Rightarrow B)$

$B_{12}$  $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$

Exercise 7

Complete the proof in Example 7 by providing comments how each step of the proof was obtained.
Exercise 7

Solution

\( B_1 \) \((A \Rightarrow B)\)
Hypothesis

\( B_2 \) \((\neg A \Rightarrow B)\)
Hypothesis

\( B_3 \) \(((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))\)
Example 5

\( B_4 \) \((\neg B \Rightarrow \neg A)\)
\( B_1, B_3 \) and MP

\( B_5 \) \(((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg \neg A))\)
Example 5 for \( A = \neg A, B = B \)

\( B_6 \) \((\neg B \Rightarrow \neg \neg A)\)
\( B_2, B_5 \) and MP
Exercise 7

\[ B_7 \quad ((\neg B \Rightarrow \neg\neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B)) \]

Axiom A3 for \( B = B, A = \neg A \)

\[ B_8 \quad ((\neg B \Rightarrow \neg A) \Rightarrow B) \]

\( B_6, B_7 \) and MP

\[ B_9 \quad B \]

\( B_4, B_8 \) and MP

\[ B_{10} \quad (A \Rightarrow B), (\neg A \Rightarrow B) \vdash B \]

\( B1 - B9 \)

\[ B_{11} \quad (A \Rightarrow B) \vdash ((\neg A \Rightarrow B) \Rightarrow B) \]

Deduction Theorem

\[ B_{12} \quad ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)) \]

Deduction Theorem
Exercise 8

Example 8
Here are consecutive steps $B_1, ..., B_3$
in a proof of $((\neg A \Rightarrow A) \Rightarrow A)$

$B_1$ $((\neg A \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow A) \Rightarrow A)))$
$B_1$ $(\neg A \Rightarrow \neg A)$
$B_1$ $((\neg A \Rightarrow A) \Rightarrow A))$

Exercise 8
Complete the proof of example 8 by providing comments how each step of the proof was obtained.

Solution

$B_1$ $((\neg A \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow A) \Rightarrow A)))$
Axiom A3 for $B = A$
$B_1$ $(\neg A \Rightarrow \neg A)$
Already proved $(A \Rightarrow A)$ for $A = \neg A$
$B_1$ $((\neg A \Rightarrow A) \Rightarrow A))$
$B_1, B_2$ and MP
We summarize all the formal proofs in $H_2$ provided in our Examples and Exercises in a form of a following Lemma

Lemma

The following formulas are provable in $H_2$

1. $(A \Rightarrow A)$
2. $(\neg\neg B \Rightarrow B)$
3. $(B \Rightarrow \neg\neg B)$
4. $(\neg A \Rightarrow (A \Rightarrow B))$
5. $((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$
6. $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$
7. $(A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B))))$
8. $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$
9. $((\neg A \Rightarrow A) \Rightarrow A)$
Proof of Completeness Theorem

**Formulas 1, 3, 4, and 7-9** from the set of provable formulas from the Lemma are all formulas we need together with $H_2$ axioms to **execute two proofs** of the **Completeness Theorem** for $H_2$.

We present these proofs in Lecture 5a and Lecture 5b. The two proofs represent two different **methods of proving** the **Completeness Theorem**.