CSE581
Computer Science Fundamentals: Theory

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P1 LOGIC: LECTURE 4a

Chapter 4 Review

PART 1: DEFINITIONS

PART 2: Problems

PART 1: Definitions from Chapter 4 you have to know

Definition: Proof System

Definition 1

By a **proof system** we understand a quadruple

$$S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$$

where

$\mathcal{L} = \{\mathcal{A}, \mathcal{F}\}$ is a **language** of S with a set $\mathcal{F}$ of formulas

$\mathcal{E}$ is a set of **expressions** of S

In particular case $\mathcal{E} = \mathcal{F}$

$LA \subseteq \mathcal{E}$ is a **non-empty, finite** set of logical axioms of S

$\mathcal{R}$ is a **non-empty, finite set** of rules of inference of S

Definition: Sound Rule of Inference

Definition 2

An inference rule

$$(r) \quad \frac{P_1 \ ; \ P_2 \ ; \ .... \ ; \ P_m}{C}$$

is sound under a semantics **M** if and only if all **M** - models of the set $\{P_1, P_2, .P_m\}$ of its **premisses** are also **M** - models of its **conclusion** C

In particular, in case of **extensional propositional semantics** when the condition below holds for any truth assignment $v : VAR \longrightarrow LV$

If $v \models_{\mathbf{M}} \{P_1, P_2, .P_m\}$, **then** $v \models_{\mathbf{M}} C$

Definition: Direct Consequence

Definition 3

For any rule of inference $r \in \mathcal{R}$ of the form

$$(r) \qquad \frac{P_1 \;\; ; \;\; P_2 \;\; ; \;\; .... \;\; ; \;\; P_m}{C}$$

C is called a **direct consequence** of $P_1, ...P_m$ by virtue of the rule $r \in \mathcal{R}$

Definition: Formal Proof

Definition 4

A formal proof of an expression $E \in \mathcal{E}$ in a proof system $S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$ is a sequence

$$A_1, A_2, , A_n \quad \text{for} \quad n \geq 1$$

of expressions from $\mathcal{E}$, such that

$$A_1 \in LA, \qquad A_n = E$$

and for each $1 < i \leq n$, either $A_i \in LA$ or A_i is a **direct consequence** of some of the **preceding expressions** by virtue of **one** of the rules of inference

$n \geq 1$ is the **length** of the proof $A_1, A_2, , A_n$

NOTATION: Provable Expressions

Notation

We write $\vdash_S E$ to denote that $E \in \mathcal{E}$ **has a formal proof** in the proof system S

A set

$$\mathbf{P}_S = \{E \in \mathcal{E} : \ \vdash_S E\}$$

is called the set of **all provable expressions** in S

Definition: Sound S

Definition 5

Given a proof system

$$S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$$

We say that the system S is **sound** under a semantics **M** iff the following conditions hold

1. Logical axioms LA are **tautologies** of under the semantics **M**, i.e.

$$LA \subseteq \mathbf{T_M}$$

2. Each **rule of inference** $r \in \mathcal{R}$ is **sound** under the semantics **M**

THEOREMS: Soundness Theorem

Let $\mathbf{P}_S$ be the set of all provable expressions of S i.e.

$$\mathbf{P}_S = \{A \in \mathcal{E} : \ \vdash_S A\}$$

Let $\mathbf{T_M}$ be a set of all expressions of S that are **tautologies** under a semantics **M**, i.e.

$$\mathbf{T_M} = \{A \in \mathcal{E} : \ \models_{\mathbf{M}} A\}$$

Our GOAL is to prove the following theorems:

Soundness Theorem (for S and semantics **M**)

$$\mathbf{P}_S \subseteq \mathbf{T_M}$$

i.e. for any $A \in \mathcal{E}$, the following implication holds

If $\vdash_S A$ **then** $\models_{\mathbf{M}} A$

THEOREMS: Completeness Theorem

Completeness Theorem (for S and semantics **M**)

$$\mathbf{P}_S = \mathbf{T}_{\mathbf{M}}$$

i.e. for any $A \in \mathcal{E}$, the following holds

$$\vdash_S A \quad \text{if and only if} \quad \models_{\mathbf{M}} A$$

The **Completeness Theorem** consists of two parts:

Part 1: **Soundness Theorem**

$$\mathbf{P}_S \subseteq \mathbf{T}_{\mathbf{M}}$$

Part 2: **Completeness Part** of the Completeness Theorem

$$\mathbf{T}_{\mathbf{M}} \subseteq \mathbf{P}_S$$

PART 2: Simple Problems

Problem 1

Given a proof system:

$$S = (\mathcal{L}_{\{\neg,\Rightarrow\}},\ \mathcal{F},\ \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\},\ \mathcal{R} = \{(r)\}$$

where $(r)\ \dfrac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))})$

Write a **formal proof** in S with 2 applications of the rule (r)

Solution: There are many solutions. Here is one of them.

Required formal proof is a sequence A_1, A_2, A_3, where

$A_1 = (A \Rightarrow A)$
(Axiom)

$A_2 = (A \Rightarrow (A \Rightarrow A))$
Rule (r) application 1 for $A = A,\ B = A$

$A_3 = ((A \Rightarrow A) \Rightarrow (A \Rightarrow (A \Rightarrow A)))$
Rule (r) application 2 for $A = A, B = (A \Rightarrow A)$

Given a proof system:

$$S = (\mathcal{L}_{\{\neg,\Rightarrow\}}, \mathcal{F}, \ \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \ \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))})$$

Problem 2

Prove that S is **sound** under classical semantics.

Solution

1. Both axioms of S are basic classical tautologies

2. Consider the rule of inference of S

$$(r) \ \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}$$

Assume that its premise (the only premise) is true, i.e. let v be any truth assignment, such that $v^*(A \Rightarrow B) = T$

We evaluate logical value of the conclusion under the truth assignment v as follows

$$v^*(B \Rightarrow (A \Rightarrow B)) = v^*(B) \Rightarrow T = T$$

for any B and any value of $v^*(B)$

Given a proof system:

$$S = (\mathcal{L}_{\{\neg,\Rightarrow\}}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))})$$

Problem 3.

Write a **formal proof** of your choice in S with 2 applications of the rule (r)

Solution

There many of such proofs, of different length, with different choice if axioms - here is my choice: A_1, A_2, A_3, where

$A_1 = (A \Rightarrow A)$

(Axiom)

$A_2 = (A \Rightarrow (A \Rightarrow A))$

Rule (r) application 1 for $A = A,\ B = A$

$A_3 = ((A \Rightarrow A) \Rightarrow (A \Rightarrow (A \Rightarrow A)))$

Rule (r) application 2 for $A = A, B = (A \Rightarrow A)$

Formal Proof

Given a proof system:

$$S = (\mathcal{L}_{\{\neg, \Rightarrow\}}, \mathcal{F}, \ \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \ \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))})$$

Problem 4

1. Prove, by constructing a **formal proof** that

$$\vdash_S ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B)))$$

Solution Required formal proof is a sequence A_1, A_2, where

$A_1 = (A \Rightarrow (\neg A \Rightarrow B))$

Axiom

$A_2 = ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B)))$

Rule (r) application for $A = A, B = (\neg A \Rightarrow B)$

Soundness Theorem

2. Does above point **1.** prove that

$$\models ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B)))?$$

Solution

Yes, it does because the system S is **sound** and we proved by Mathematical Induction over the length of a proof that if S is **sound**, then the **Soundness Theorem** holds for S

Soundness

Problem 5

Given a proof system:

$$S = (\mathcal{L}_{\{\neg,\Rightarrow\}},\ \mathcal{F},\ \ \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\},\ \ (r)\ \frac{(A\Rightarrow B)}{(B\Rightarrow(A\Rightarrow B))})$$

Prove that S is **not sound** under **K** semantics

Solution

Axiom $(A \Rightarrow A)$ is not a **K** semantics tautology

Any truth assignment v such that $v^*(A) = \perp$ is a **counter-model** for it

This proves that S is **not sound** under **K** semantics