CSE581
Computer Science Fundamentals: Theory

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P1 LOGIC: LECTURE 4

Chapter 4
GENERAL PROOF SYSTEMS

PART 1: General Introduction

Proof Systems - Intuitive Definition

Proof systems are built to prove, it means to **construct formal proofs** of statements formulated in a given language

First component of any proof system is hence its **formal language** $\mathcal{L}$

Proof systems are inference machines with statements called **provable statements** being their final products

Semantical Link

The **starting points** of the inference machine of a proof system S are called its **axioms**

We distinguish two kinds of axioms: **logical axioms** LA and **specific axioms** SA

Semantical link: we usually build a proof systems for a given language and its **semantics** i.e. for

a logic defined semantically

Semantical Link

We always choose as a set of **logical axioms** LA some **subset of tautologies**, under a given **semantics**

We will **consider here** only proof systems with **finite sets** of **logical** or **specific axioms** , i.e we will examine only **finitely axiomatizable** proof systems

Semantical Link

We can, and we often do, consider proof systems with languages **without yet established semantics**

In this case the **logical axioms LA** serve as description of **tautologies** under a future **semantics** yet to be built

Logical axioms LA of a proof system S are hence not only tautologies under an established **semantics**, but they can also guide us how to define a semantics when it is yet **unknown**

Specific Axioms

The **specific axioms SA** consist of statements that describe a specific knowledge of an universe we want to use the proof system S to prove facts about

Specific axioms SA are not universally true

Specific axioms SA are true only in the universe we are interested to **describe** and **investigate** by the use of the proof system S

Formal Theory

Given a **proof system** S with **logical axioms** LA

Specific axioms SA of the proof system S is any finite set of formulas that are not **tautologies**, and hence they are always disjoint with the set of **logical axioms LA** of S

The **proof system** S with added set of **specific axioms SA** is called a **formal theory** based on S

Inference Machine

The **inference machine** of a proof system S is defined by a finite set of **inference rules**

The **inference rules** describe the way we are allowed to **transform** the information within the system with **axioms** as a staring point

We depict it informally on the next slide

Inference Machine

AXIOMS

↓ ↓ ↓

RULES applied to AXIOMS

↓ ↓ ↓

RULES applied to any expressions above

↓ ↓ ↓

Provable formulas

Semantical Link

Semantical link:

Rules of inference of a system S have to preserve the truthfulness of what they are being used to prove

The notion of truthfulness is always defined by a given semantics **M**

Rules of inference that preserve the truthfulness are called **sound rules** under a given semantics **M**

Rules of inference can be sound under one semantics and not sound under another

Soundness Theorem

Goal 1

When developing a proof system S the first goal is prove the following theorem about it and its semantics **M**

Soundness Theorem

For any formula A of the language of the system S

If a formula A is **provable** from **logical axioms** LA of S only, then A is a **tautology** under the semantics **M**

Propositional Proof Systems

We discuss here first only proof systems for propositional languages and call them **proof systems** for different propositional logics

Remember

The notion of **soundness** is connected with a given **semantics**

A proof system S can be sound under **one semantics,** and not sound under the **other**

For example a set of axioms and rules sound under classical logic semantics might not be sound under Ł logic semantics, or K logic semantics, or others

Completeness of the Proof Systems

In general there are many proof systems that are sound under a given **semantics**, i.e. there are many sound proof systems for a given **logic** semantically defined

Given a proof system S with **logical axioms** LA that is **sound** under a semantics **M**.

Notation

Denote by $\mathbf{T_M}$ the set of all tautologies defined by the semantics **M**, i.e. we have that

$$\mathbf{T_M} = \{A \in \mathcal{F} : \models_{\mathbf{M}} A\}$$

Completeness Property

A **natural question** arises:

Are all tautologies i.e formulas $A \in \mathbf{T_M}$ **provable** in the system S ??

We assume that we have already proved that S is sound under the semantics **M**

The positive answer to this question is called **completeness property** of the system S .

Completeness Theorem

Goal 2

Given for a **sound** proof system S under its semantics **M**, our the second goal is to prove the following theorem about S

Completeness Theorem

For any formula A of the language of S

A **is provable** in S iff A is a **tautology** under the semantics **M**

We write the **Completeness Theorem** symbolically as

$$\vdash_S A \quad \text{iff} \quad \models_{\mathbf{M}} A$$

Completeness Theorem is composed of two parts:

Soundness Theorem and the **Completeness Part** that proves the completeness property of a sound proof system

Proving Soundness and Completeness

Proving the Soundness Theorem for S under a semantics **M** is usually a straightforward and not a very difficult task

We **first prove** that all **logical axioms** LA are **tautologies**, and then we **prove** that all inference rules of the system S preserve the notion of the truth

Proving the completeness part of the **Completeness Theorem** is always a crucial, difficult and sometimes impossible task

OUR PLAN

We will study two proofs of the **Completeness Theorem** for **classical propositional** proof system in Chapter 5

We will present a constructive proofs of **Completeness Theorem** for two different Gentzen style automated theorem proving systems for **classical Logic** in Chapter 6

We discuss the **Inuitionistic Logic** in Chapter 7

Predicate Logics are discussed Chapters 8, 9, 10, 11

PART 2
PROOF SYSTEMS: Formal Definitions

Proof System S

In this section we present **formal definitions** of the following notions

Proof system S

Formal proof from **logical axioms** in a proof system S

Formal proof from **specific axioms** in a proof system S

Formal Theory based on a proof system S

We also give **examples** of different simple proof systems

Components: Language

Language $\mathcal{L}$ of a **proof system** S is any formal language $\mathcal{L}$

$$\mathcal{L} = (\mathcal{A}, \mathcal{F})$$

We assume as before that both sets $\mathcal{A}$ and $\mathcal{F}$ are enumerable, i.e. we deal here with enumerable languages

The **Language** $\mathcal{L}$ can be **propositional** or **first order** (predicate) but we discuss propositional languages first

Components: Expressions

Expressions $\mathcal{E}$ of a proof system S

Given a set $\mathcal{F}$ of well formed formulas of the language $\mathcal{L}$ of the system S

We often extend the set $\mathcal{F}$ to some set $\mathcal{E}$ of **expressions** build out of the language $\mathcal{L}$ and some extra symbols, if needed

In this case all other components of S are also defined on basis of elements of the set of **expressions** $\mathcal{E}$

In particular, and **most common case** we have that $\mathcal{E} = \mathcal{F}$

Expressions Examples

Automated theorem proving systems usually use as their basic components different sets of **expressions** build out of formulas of the language $\mathcal{L}$

In Chapters 6 and 10 we consider finite sequences of formulas instead of formulas, as basic expressions of the proof systems **RS** and **RQ**

We also present there proof systems that use yet other kind of expressions, called original **Gentzen sequents** or their modifications

Some systems use yet other expressions such as clauses, sets of clauses, or sets of formulas, others use yet still different expressions

Semantical Link

We always have to **extend** a given semantics **M** for the language $\mathcal{L}$ of the system S to the set $\mathcal{E}$ of all **expression** of the system S

Sometimes, like in case of **Resolution** based proof systems we have also to **prove** a semantic equivalency of new created expressions $\mathcal{E}$ (sets of clauses in Resolution case) with appropriate formulas of $\mathcal{L}$

Example

For example, in the automated theorem proving system **RS** presented in Chapter 6 the basic expressions $\mathcal{E}$ are **finite sequences** of formulas of $\mathcal{L} = \mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$.

We **extend** our classical semantics for $\mathcal{L}$ to the set $\mathcal{F}^*$ of all **finite sequences** of formulas as follows:

For any $v : VAR \longrightarrow \{F, T\}$ and

any $\Delta \in \mathcal{F}^*$, $\Delta = A_1, A_2, ..A_n$, we put

$$v^*(\Delta) = v^*(A_1, A_2, ..A_n)$$

$$= v^*(A_1) \cup v^*(A_2) \cup .... \cup v^*(A_n)$$

i.e. in a shorthand notation

$$\Delta \equiv (A_1 \cup A_2 \cup ... \cup A_n)$$

Components: Logical Axioms

Logical axioms LA of S form a **non-empty** subset of the set $\mathcal{E}$ of **expressions** of the proof system S, i.e.

$$LA \subseteq \mathcal{E}$$

In particular, LA is a non-empty subset of **formulas**, i.e.

$$LA \subseteq \mathcal{F}$$

We **assume here** that the set LA of **logical axioms** is always **finite** , i.e. that we consider here finitely axiomatizable systems

In general, **we assume** that the set LA is primitively recursive i.e. that there is an effective procedure to determine whether a given expression $E \in \mathcal{E}$ **is** or **is not** in AL

Components: Axioms

Semantical link

Given a semantics **M** for $\mathcal{L}$ and its **extension** to the set $\mathcal{E}$ of all expressions

We extend the notion of **tautology** to the expressions and write

$$\models_{\mathbf{M}} E$$

to denote that the **expression** $E \in \mathcal{E}$ is a **tautology** under semantics **M** and we put

$$\mathbf{T_M} = \{E \in \mathcal{E} : \models_{\mathbf{M}} E\}$$

Logical axioms LA are always a subset of expressions that are **tautologies** of under the semantics **M**, i.e.

$$LA \subseteq \mathbf{T_M}$$

Components: Rules of Inference

Rules of inference $\mathcal{R}$

We **assume** that a proof system contains only a finite number of **inference rules**

We **assume** that each rule has a finite number of **premisses** and one **conclusion**

We also **assume** that one can effectively decide, for any **inference rule**, whether a given string of expressions **form** its premisses and conclusion or **do not**, i.e. that

All rules $r \in \mathcal{R}$ are primitively recursive

Components: Rules of Inference

Definition

Each **rule of inference** $r \in \mathcal{R}$ is a **relation** defined in the set $\mathcal{E}^m$, where $m \geq 1$ with values in $\mathcal{E}$, i.e.

$$r \subseteq \mathcal{E}^m \times \mathcal{E}$$

Elements $P_1, P_2, \ldots P_m$ of a tuple $(P_1, P_2, \ldots P_m, C) \in r$ are called **premisses** of the rule r and C is called its **conclusion**

All $r \in \mathcal{R}$ are **primitively recursive** relations

Components: Rules of Inference

We write the **inference rules** in a following convenient way

One premiss rule

$$(r) \quad \frac{P_1}{C}$$

Two premisses rule

$$(r) \quad \frac{P_1 \ ; \ P_2}{C}$$

m premisses rule

$$(r) \quad \frac{P_1 \ ; \ P_2 \ ; \ .... \ ; \ P_m}{C}$$

Semantic Link: Sound Rules of Inference

Given some **m** premisses rule

$$(r) \quad \frac{P_1 \;;\; P_2 \;;\; .... \;;\; P_m}{C}$$

Semantical link

Given a semantics **M** for the language $\mathcal{L}$ and for the set of expressions $\mathcal{E}$

We want the **rules of inference** $r \in \mathcal{R}$ to preserve truthfulness i.e. to be **sound** under the semantics **M**

General Definition: Sound Rule of Inference

Definition

Given an inference rule $r \in \mathcal{R}$

$$(r) \quad \frac{P_1 \; ; \; P_2 \; ; \; .... \; ; \; P_m}{C}$$

We say that the inference rule $r \in \mathcal{R}$ is **sound** under a semantics **M**

if and only if

all **M** - models of the set $\{P_1, P_2, .P_m\}$ of its **premisses** are also **M** - models of its **conclusion** C

Propositional Definition: Sound Rule of Inference

In propositional languages case, the semantics **M**, and hence the **M** - models are defined in terms of the truth assignment $v : VAR \longrightarrow LV$, where LV is the set of logical values for the semantics **M**

Definition

An inference rule $r \in \mathcal{R}$, such that

$$(r) \quad \frac{P_1 \;;\; P_2 \;;\; .... \;;\; P_m}{C}$$

is sound under a semantics **M**

if and only if

the condition below holds or any $v : VAR \longrightarrow LV$

If $v \models_{\mathbf{M}} \{P_1, P_2, .P_m\}$, **then** $v \models_{\mathbf{M}} C$

Example

Given a rule of inference

$$(r) \quad \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}$$

Prove that (r) is **sound** under classical semantics

Let v be any truth assignment, such that $v \models (A \Rightarrow B)$, i.e. by definition $v^*(A \Rightarrow B) = T$

We evaluate logical value of the **conclusion** under v as follows

$$v^*(B \Rightarrow (A \Rightarrow B)) = v^*(B) \Rightarrow T = T$$

for any B and any value of $v^*(B)$

This proves that $v \models (B \Rightarrow (A \Rightarrow B))$ and hence the **soundness** of (r)

Formal Definition: Proof System

Definition

By a **proof system** we understand a quadruple

$$S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$$

where

$\mathcal{L} = \{\mathcal{A}, \mathcal{F}\}$ is a **language** of S with a set $\mathcal{F}$ of formulas

$\mathcal{E}$ is a set of **expressions** of S

In particular case $\mathcal{E} = \mathcal{F}$

$LA \subseteq \mathcal{E}$ is a **non- empty, finite set** of logical axioms of S

$\mathcal{R}$ is a **non- empty, finite set** of rules of inference of S

PART 3: Formal Proofs
Simple Examples of Proof Systems

Provable Expressions

A **final product** of a single or multiple use of the **inference rules** of S, with axioms taken as a **starting point** are called **provable expressions** of the proof system S

A single use of an inference rule is called a **direct consequence**

A multiple application of rules of inference with axioms taken as a starting point is called a **proof**

Definition: Direct Consequence

Formal definitions are as follows

Direct consequence

For any rule of inference $r \in \mathcal{R}$ of the form

$$(r) \quad \frac{P_1 \ ; \ P_2 \ ; \ .... \ ; \ P_m}{C}$$

C is called a **direct consequence** of $P_1, ...P_m$ by virtue of the rule $r \in \mathcal{R}$

Definition: Formal Proof

Formal Proof of an expression $E \in \mathcal{E}$ in a proof system

$$S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$$

is a sequence

$$A_1, A_2, , A_n \quad \text{for} \quad n \geq 1$$

of expressions from $\mathcal{E}$, such that

$$A_1 \in LA, \qquad A_n = E$$

and for each $1 < i \leq n$, either $A_i \in LA$ or A_i is a **direct consequence** of some of the **preceding expressions** by virtue of one of the rules of inference

$n \geq 1$ is the **length of the proof** $A_1, A_2, , A_n$

Formal Proof Notation

We write

$$\vdash_S E$$

to denote that $E \in \mathcal{E}$ **has a proof** in S

When the proof system S is **fixed** we write $\vdash E$

Any $E \in \mathcal{E}$, such that $\vdash_S E$ is called a **provable expression** of S

The set of **all provable expressions** of S is denoted by $\mathbf{P}_S$, i.e. we put

$$\mathbf{P}_S = \{E \in \mathcal{E} : \vdash_S E\}$$

PART 4: Hypothesis, Consequence, Soundness and Completeness

Proof from Hypothesis

While proving expressions we often use some extra information available, besides the axioms of the proof system.

This extra information is called hypothesis in the proof.

Let $\Gamma \subseteq \mathcal{E}$ be a set expressions called hypothesis

A proof of $E \in \mathcal{E}$ **from the set of** hypothesis Γ in S is **a formal proof** in S, where the expressions from Γ are treated as **additional** hypothesis **added** to the set LA of the **logical axioms** of the system S

Notation: $\Gamma \vdash_S A$

We read it : A has a proof in S from the set Γ (and logical axioms LA)

Definition: Proof from Hypothesis

Definition

We say that A **has a proof** in S **from the set** Γ (and logical axioms LA) if and only if

there is a sequence $A_1, ... A_n$ of expressions from $\mathcal{E}$, such that

$$A_1 \in LA \cup \Gamma, \quad A_n = A$$

and for each $1 < i \leq n$, either $A_i \in LA \cup \Gamma$ or A_i is a **direct consequence** of some of the preceding expressions by virtue of one of the rules of inference

We denote it as $\Gamma \vdash_S A$

Special Cases

We usually consider and use the case when the set of hypothesis is finite.

Case of $\Gamma \subseteq \mathcal{E}$ **finite set** and $\Gamma = \{B_1, B_2, ..., B_n\}$

We use **notation**

$$B_1, B_2, ..., B_n \vdash_{\mathcal{S}} A$$

for $\{B_1, B_2, ..., B_n\} \vdash_{\mathcal{S}} A$

Case of $\Gamma = \emptyset$ is also a special one.

By the definition of a proof of A from Γ, $\emptyset \vdash A$ means that in the proof of A we use only axioms LA of S

We hence use **notation** $\vdash_S A$

to denote that A has a proof from empty Γ; i.e. A has a proof from logical axioms only

Definition: Consequences of Γ

Definition

For any $\Gamma \subseteq \mathcal{E}$, and $A \in \mathcal{E}$,

If $\Gamma \vdash_S A$, then A is called a **consequence** of Γ in S

Definition

We denote by $\mathbf{Cn}_S(\Gamma)$ the **set of all consequences** of Γ in S, i.e. we put

$$\mathbf{Cn}_S(\Gamma) = \{A \in \mathcal{E} : \quad \Gamma \vdash_S A\}$$

Definition: Consequence Operation

Observe that by defining a consequence of Γ in S, we define in fact a **function** which to every set $\Gamma \subseteq \mathcal{E}$ assigns a set of **all its consequences** $\mathbf{Cn}_S(\Gamma)$

We denote this function by $\mathbf{Cn}_S$ and adopt the following

Definition

Any function

$$\mathbf{Cn}_S : 2^{\mathcal{E}} \longrightarrow 2^{\mathcal{E}}$$

such that for every $\Gamma \in 2^{\mathcal{E}}$

$$\mathbf{Cn}_S(\Gamma) = \{A \in \mathcal{E} : \quad \Gamma \vdash_S A\}$$

is called the **consequence operation** in S

Consequence Operation: Monotonicity

Take any **consequence operation**

$$\mathbf{Cn}_S : 2^{\mathcal{E}} \longrightarrow 2^{\mathcal{E}}$$

Monotonicity Property

For any sets Γ, Δ of expressions of S,

if $\Gamma \subseteq \Delta$ **then** $\mathbf{Cn}_S(\Gamma) \subseteq \mathbf{Cn}_S(\Delta)$

Exercise: write the proof;

it follows directly from the definition of $\mathbf{Cn}_S$ and definition of the formal proof

Consequence Operation: Transitivity

Take any **consequence operation**

$$\mathbf{Cn}_S : 2^{\mathcal{E}} \longrightarrow 2^{\mathcal{E}}$$

Transitivity Property

For any sets $\Gamma_1, \Gamma_2, \Gamma_3$ of expressions of S,

if $\Gamma_1 \subseteq \mathbf{Cn}_S(\Gamma_2)$ and $\Gamma_2 \subseteq \mathbf{Cn}_S(\Gamma_3)$, **then** $\Gamma_1 \subseteq \mathbf{Cn}_S(\Gamma_3)$

Exercise: write the proof;

it follows directly from the definition of $\mathbf{Cn}_S$ and definition of the formal proof

Consequence Operation: Finiteness

Take any **consequence operation**

$$\mathbf{Cn}_S : 2^{\mathcal{E}} \longrightarrow 2^{\mathcal{E}}$$

Finiteness Property

For any expression $A \in \mathcal{E}$ and any set $\Gamma \subseteq \mathcal{E}$,
$A \in \mathbf{Cn}_S(\Gamma)$ if and only if there is a **finite subset** Γ_0 of Γ such that $A \in \mathbf{Cn}_S(\Gamma_0)$

Exercise: write the proof;

it follows directly from the definition of $\mathbf{Cn}_S$ and definition of the formal proof

Definition: Sound S

Definition

Given a proof system

$$S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$$

We say that the system S is **sound** under a semantics **M** iff the following conditions hold

1. $LA \subseteq \mathbf{T_M}$
2. Each rule of inference $r \in \mathcal{R}$ is **sound**

Example

Given a proof system:

$$S = (\mathcal{L}_{\{\neg,\Rightarrow\}},\ \mathcal{F},\ \ \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\},\ \ (r)\ \frac{(A\Rightarrow B)}{(B\Rightarrow(A\Rightarrow B))})$$

1. Prove that S is **sound** under **classical semantics**
2. Prove that S is **not sound** under **K** semantics

Example

1. Both **axioms** of S are basic classical **tautologies** and we have just proved that the rule of inference (r) is **sound**, hence S is **sound**

2. Axiom $(A \Rightarrow A)$ is not a **K** semantics tautology

Any truth assignment v such that $v^*(A) = \perp$ is a **counter-model** for it

This proves that S is **not sound** under **K** semantics

Soundness Theorem

Let $\mathbf{P}_S$ be the set of all provable expressions of S i.e.

$$\mathbf{P}_S = \{A \in \mathcal{E} : \ \vdash_S A\}$$

Let $\mathbf{T_M}$ be a set of all expressions of S that are tautologies under a semantics **M**, i.e.

$$\mathbf{T_M} = \{A \in \mathcal{E} : \ \models_{\mathbf{M}} A\}$$

Soundness Theorem for S and semantics **M**

$$\mathbf{P}_S \subseteq \mathbf{T_M}$$

i.e. for any $A \in \mathcal{E}$, the following implication holds

If $\vdash_S A$, then $\models_{\mathbf{M}} A$.

Exercise: prove by Mathematical Induction over the length of a proof that if S is sound, the Soundness Theorem holds for S

Completeness Theorem

Completeness Theorem for S and semantics **M**

$$\mathbf{P}_S = \mathbf{T}_\mathbf{M}$$

i.e. for any $A \in \mathcal{E}$, the following holds

$$\vdash_S A \quad \text{if and only if} \quad \models_\mathbf{M} A$$

The **Completeness Theorem** consists of two parts:

Part 1: **Soundness Theorem**

$$\mathbf{P}_S \subseteq \mathbf{T}_\mathbf{M}$$

Part 2: **Completeness Part** of the Completeness Theorem

$$\mathbf{T}_\mathbf{M} \subseteq \mathbf{P}_S$$

Syntactic Consistency: Formal Theories

Formal theories play crucial role in mathematics and were historically defined for classical **predicate (first order)** logic and consequently for other non-classical logics

They are routinely called **first order theories**

We discuss them in detail in Chapter 10 dealing formally with classical predicate logic

First order theories are hence based on a proof systems S with a predicate (first order) language $\mathcal{L}$

We sometimes consider **formal theories** based on proof systems with a propositional language $\mathcal{L}$ and we call them **propositional theories**

Syntactic Consistency: Formal Theories

Given a proof system $S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$

We build (define) a **formal theory** based on S as follows.

1. We **select** a certain **finite** subset SA of expressions of S, **disjoint** with the logical axioms LA of S

The set *SA* is called a set of **specific** axioms of the **formal theory** based on S

2. We use set SA of **specific** axioms to define a language $\mathcal{L}_{SA}$, called a **language** of the formal theory

Here we have two cases

Syntactic Consistency: Formal Theories

c1 S is a first order proof system, i.e. $\mathcal{L}$ of S is a **predicate** language

We **define** the language $\mathcal{L}_{SA}$ by **restricting** the sets of constant, functional, and predicate symbols of $\mathcal{L}$ to constant, functional, predicate symbols **appearing** in the set SA of **specific axioms**

Both languages $\mathcal{L}_{SA}$ and $\mathcal{L}$ **share** the same set of propositional connectives

c2 S is a **propositional** proof system, i.e. $\mathcal{L}$ of S is a **propositional** language $\mathcal{L}_{SA}$ is defined by **restricting** $\mathcal{L}$ to connectives appearing in the set SA

Syntactic Consistency: Formal Theories

Definition

Given a proof system $S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$ and **finite** subset SA of expressions of S, **disjoint** with the logical axioms LA

The system

$$T = (\mathcal{L}, \mathcal{E}, LA, SA, \mathcal{R})$$

is called a **formal theory** based on S

The set SA is the set of **specific axioms** of T

The language $\mathcal{L}_{SA}$ defined by **c1** or **c2** is called the language of the **theory** T

Syntactic Consistency

Definition

A theory

$$T = (\mathcal{L}, \mathcal{E}, LA, SA, \mathcal{R})$$

is **consistent** if and only if there exists an expression $E \in \mathcal{E}_{SA}$ such that $E \notin \mathbf{T}(SA)$, i.e. such that

$$SA \nvdash_S E$$

otherwise the theory T is **inconsistent**.

Observe that the definition has purely syntactic meaning

Syntactic Consistency: Formal Theories

The **consistency** definition reflexes our intuition what proper notion of **provability** should mean

Namely, it says that a formal **theory** T based on a proof system S is **consistent** only when it **does not prove** all expressions (formulas in particular cases) of $\mathcal{L}_{SA}$

The **theory** T such that it **proves everything** stated in $\mathcal{L}_{SA}$ obviously should be, and is defined as **inconsistent**

Syntactic Consistency: Formal Theories

In particular, we have the following **syntactic definition** of consistency and inconsistency for any proof system S

Definition

A proof system

$$S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$$

is **consistent** if and only if there exists $E \in \mathcal{E}$ such that $E \notin \mathbf{P}_S$, i.e. such that

$$\nvdash_S E$$

otherwise S is **inconsistent**

Formal Theory

Given a proof system $S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$. Let a set $SA \subseteq \mathcal{E}$ be such that

$$SA \cap \mathbf{T_M} = \emptyset$$

A **formal theory** with the set of specific axioms SA is denoted by $T(SA)$ and defined as follows

$$T(SA) = (\mathcal{L}_{SA}, \mathcal{E}, LA, SA, \mathcal{R})$$

The set of all expressions of the language $\mathcal{L}_{SA}$ provable from the set specific axioms SA (and logical axioms LA) i.e. the set

$$\mathbf{T}(SA) = \{A \in \mathcal{E} : \quad SA \vdash_S A\}$$

is called the set of all **theorems** of the theory $T(SA)$

Soundness of the Theory

Soundness Theorem for a formal theory $T(SA)$ based on a proof system S says:

For any formula A of the language $\mathcal{L}_{SA}$ of the theory $T(SA)$,
if a formula A is **provable** in the theory $T(SA)$,
then A is **true** in any **model** of the set of specific axioms SA of $T(SA)$

Syntactic Completeness of Formal Theories

The **Completeness Theorem** for the proof system S established equivalency of the notion of provability and tautology:

$$\mathbf{P}_S = \mathbf{T}_{\mathbf{M}}$$

Observe the equation $\mathbf{P}_S = \mathbf{T}_{\mathbf{M}}$ holds for a theorie $T(SA)$ **only when** the set of its specific axioms $SA = \emptyset$

We nevertheless talk about **Complete/Incomplete** theories as the final goal of the course (and the book) is to prove the **Gödel Incompleteness Theorem** for the Peano formal theory of the Arithmetic of Natural Numbers

Complete Formal Theory

Definition

A **formal theory** $T(SA)$ based on a language with negation $\neg$ is **complete** if and only if **for any** A of the language of the theory $T(SA)$ the following holds

$$A \in \mathbf{T}(SA) \quad \text{or} \quad \neg A \in \mathbf{T}(SA)$$

Otherwise a theory $T(SA)$ is **incomplete**

The completeness of a theory means that we can **prove** or **disapprove** any statement formulated within it

It hence corresponds to the natural meaning of the notion of a complete information

Syntactic Consistence

Definition

A formal theory $T(SA)$ based on a language with negation $\neg$ is **consistent** if and only if **there is no** formula A of the language of the theory $T(SA)$ such that

$$A \in \mathbf{T}(SA) \quad \text{and} \quad \neg A \in \mathbf{T}(SA)$$

Otherwise $T(SA)$ is **inconsistent**

The notions of consistency, inconsistency and completeness, incompleteness describe are the most important properties of any theory

PART 5: Decidable and Syntactically Decidable Proof Systems

Decidable and Syntactically Decidable Proof Systems

A proof system S is called **decidable** when there is a finite, mechanical method for determining, given any expression $A \in \mathcal{E}$ whether there is a proof of A in S; i.e. whether $A \in \mathbf{P}_S$

otherwise S is called **undecidable**

Observe that the above notion of decidability of the system does not require to find a proof

It requires only a mechanical procedure of deciding whether a proof exists for any expression of the system.

Example

We **prove now** that A Hilbert style proof system S for classical propositional logic presented in Chapter 9 is decidable

We first prove the Completeness Theorem for it

$$\mathbf{P}_S = \mathbf{T_M}$$

We get that for any $A \in \mathcal{E}$

$$A \notin \mathbf{P}_S \quad \text{iff} \quad A \notin \mathbf{T_M}$$

We have proved already that that the notion of classical propositional tautology, i.e. the statement $A \notin \mathbf{T_M}$ is decidable

We conclude: the system S is decidable

Syntactically Decidable Systems

A proof system S is **syntactically decidable** if it is possible to define for it a finite, mechanical method that generates a proof for any given expression A of S

otherwise the system S **is not not syntactically decidable**

We call such syntactically decidable systems **automated theorem proving** systems

Syntactically Decidable Systems

All Gentzen type proof systems presented here are both decidable or semi-decidable and syntactically decidable or syntactically semi-decidable.

We usually call them **automated theorem proving** systems for different logics under consideration.

Resolution based proof systems are also wildly known examples of the syntactically decidable, or semi-decidable systems.

Finding a Gentzen Type, or Resolution type formalization for a given logic is a standard question one asks about any logic being developed.

Formal Proofs

Remember that the notion of a formal proof in a system S is purely syntactical in its nature

Formal Proof carries a semantical meaning via established semantics and the Soundness Theorem

The rules of inference of a proof system define only how to transform strings of symbols of the language into another string of symbols.

The formal proof, by the definition says that in order to **prove** an expression A in a system S one has to construct of a sequence of proper transformations, defined by the rules of inference.

Simple System S_1

Consider a very simple proof system system S_1 with $\mathcal{E} = \mathcal{F}$

$$S_1 = (\mathcal{L}_{\{P, \Rightarrow\}},\ \mathcal{F},\ LA1 = \{(A \Rightarrow A)\},\ (r)\ \frac{B}{PB}),$$

where $A, B \in \mathcal{F}$ are any formulas and

where P is some one argument connective;

we might read PA for example as " it is possible that A"

Observe that even the system S_1 has only one axiom, it represents an infinite number of formulas.

We call such axiom **axiom schema**

Simple System S_2

Consider now a system S_2

$$S_2 = (\mathcal{L}_{\{P,\Rightarrow\}},\ \mathcal{F}\ \ LA2 = \{(a \Rightarrow a)\}, (r)\ \frac{B}{PB}),$$

where $a \in VAR$ is any variable (atomic formula) and $B \in \mathcal{F}$ is any formula

Observe that even the system S_1 has only one axiom, it is also an **axiom schema**

Observe that for example a formula

$$((Pa \Rightarrow (b \Rightarrow c)) \Rightarrow (Pa \Rightarrow (b \Rightarrow c)))$$

is an axiom of system S_1

but is not an axiom of the system S_2

Some Provable Formulas

We have that

$$\vdash_{S_1} ((Pa \Rightarrow (b \Rightarrow c)) \Rightarrow (Pa \Rightarrow (b \Rightarrow c)))$$

because

$$((Pa \Rightarrow (b \Rightarrow c)) \Rightarrow (Pa \Rightarrow (b \Rightarrow c))) \in \text{LA1}$$

other provable formulas are

$$\vdash_{S_1} P(a \Rightarrow a), \quad \vdash_{S_2} P(a \Rightarrow a),$$

$$\vdash_{S_1} PP(a \Rightarrow a), \quad \vdash_{S_2} PP(a \Rightarrow a)$$

Formal Proofs

Formal proofs in both systems of above formulas are identical and are as follows.

Formal proof of $P(a \Rightarrow a)$ in S_1 and S_2 is:

$A_1 = (a \Rightarrow a)$, axiom

$A_2 = P(a \Rightarrow a)$ rule application for $B = (a \Rightarrow a)$

Formal Proofs

Formal proof of $PP(a \Rightarrow a)$ in S_1 and S_2 is:

$A_1 = (a \Rightarrow a)$, axiom

$A_2 = P(a \Rightarrow a)$, rule application for $B = (a \Rightarrow a)$

$A_3 = PP(a \Rightarrow a)$ rule application for $B = P(a \Rightarrow a)$

Proof Search

Let's **search for a proof** (if exists) of the formula A below in S_2

$$A = PP((Pa \Rightarrow (b \Rightarrow c)) \Rightarrow (Pa \Rightarrow (b \Rightarrow c)))$$

Observe, that if A had the proof, the only last step in this proof would be the application of the rule $(r)\ \frac{B}{PB})$ to the formula

$$P((Pa \Rightarrow (b \Rightarrow c)) \Rightarrow (Pa \Rightarrow (b \Rightarrow c)))$$

This formula, in turn, if it had the proof, the only last step in its proof would be the application of the rule r to the formula

$$((Pa \Rightarrow (b \Rightarrow c)) \Rightarrow (Pa \Rightarrow (b \Rightarrow c)))$$

The **search process stops here**

Proof Search

Observe that

$((Pa \Rightarrow (b \Rightarrow c)) \Rightarrow (Pa \Rightarrow (b \Rightarrow c))) \notin LA2$

what means that our **search** for the proof has **failed**;

i.e. our found sequence of formulas does not constitute a proof

Moreover, the search was, at each step unique what proves that the proof of A in S_2 **does not exist**, i.e.

$$\nvdash_{S_2} \quad PP((Pa \Rightarrow (b \Rightarrow c)) \Rightarrow (Pa \Rightarrow (b \Rightarrow c)))$$

Proof Search Procedure

We easily **generalize** above example to a proof search procedure to any formula A of $S1$ or $S2$ as follows

Procedure SP

Step: Check the main connective of A

If main connective is P, it means that A was obtained by the rule r

Erase the main connective P

Repeat until no P as a main connective is l eft.

If the main connective is $\Rightarrow$ check if a formula is an axiom

If it is an axiom, **stop** and **yes** we have a proof

If it is not an axiom, **stop** and **no**, proof does not exist

Syntactical Decidability

The **Procedure SP** is a finite, effective, automatic procedure of searching for a proof of formulas in both our proof systems.

This proves the following.

Fact Proof systems S_1 and S_2 are syntactically decidable

Semantical link

Remark that we haven't defined a semantics for the language $\mathcal{L}_{\{\Rightarrow, P\}}$ of systems $S1, S2$

We can't talk about the soundness of these systems yet

but we can think how to define a sound semantics for our systems.

If we want to understand statement PA as " A is possible" we need to define some kind of **modal** semantics.

Semantical link

All known modal semantics **extend** the classical semantics, i.e. they are the same as classical one on non-modal connectives

Hence under any possible modal semantics axioms $S1, S2$ of would be a sound axiom under standard modal logics semantics, as they are classical tautologies.

To assure the soundness of both systems we must have a modal semantics **M** that makes the rule

$$(r)\ \frac{B}{PB}$$

sound under the modal semantics **M**

General Question 1

General Q1: Are all proof systems decidable?

Answer Q1: No, not all proof systems are decidable

The most ”natural” and historically first developed proof system for classical predicate logic is **not decidable**

General Question 2

General Q2 Can we give an example of a logic and its complete proof system which is not decidable, but the logic does have another complete, syntactically decidable proof system?

Answer Q2: Hilbert style proof system for classical propositional logic presented in chapter 5 is complete and decidable but is not syntactically decidable

We present in chapter 6 some complete proof systems for classical propositional logic that are syntactically decidable