CSE581
Computer Science Fundamentals: Theory

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Chapter 3 REVIEW
Some Definitions and Problems
SOME DEFINITIONS: Part One

There are some basic **Definitions** and sample **Questions** with Solutions from **Chapter 3**

**Study** them for **MIDTERM**

Knowing all basic **Definitions** is the first step for understanding the material and solving Problems. **Solutions** are very carefully written - so you could understand them step by step and hence correctly write yours, which do not need to be that detailed.
**DEFINITIONS: Propositional Extensional Semantics**

**Definition 1**

Given a propositional language $\mathcal{L}_{CON}$ for the set $CON = C_1 \cup C_2$, where $C_1, C_2$ are respectively the sets of unary and binary connectives.

Let $V$ be a non-empty set of **logical values**.

**Connectives** $\nabla \in C_1$, $\circ \in C_2$ are called **extensional** iff their semantics is defined by respective functions:

$$\nabla : V \rightarrow V \quad \text{and} \quad \circ : V \times V \rightarrow V$$
DEFINITIONS: Propositional Extensional Semantics

Definition 2
Formal definition of a **propositional extensional semantics** for a given language $L_{CON}$ consists of providing **definitions** of the following four main components:

1. Logical Connectives
2. Truth Assignment
3. Satisfaction, Model, Counter-Model
4. Tautology
CLASSICAL PROPOSITIONAL SEMANTICS
DEFINITIONS: Truth Assignment Extension $v^*$

Definition 3
The Language: $\mathcal{L} = \mathcal{L}_{\neg, \Rightarrow, \cup, \cap}$

Given the truth assignment $v : \text{VAR} \rightarrow \{T, F\}$ in classical semantics for the language $\mathcal{L} = \mathcal{L}_{\neg, \Rightarrow, \cup, \cap}$

We define its extension $v^*$ to the set $\mathcal{F}$ of all formulas of $\mathcal{L}$ as $v^* : \mathcal{F} \rightarrow \{T, F\}$ such that

(i) for any $a \in \text{VAR}$

$$v^*(a) = v(a)$$

(ii) and for any $A, B \in \mathcal{F}$ we put

$$v^*(\neg A) = \neg v^*(A);$$
$$v^*((A \cap B)) = \cap (v^*(A), v^*(B));$$
$$v^*((A \cup B)) = \cup (v^*(A), v^*(B));$$
$$v^*((A \Rightarrow B)) = \Rightarrow (v^*(A), v^*(B));$$
$$v^*((A \Leftrightarrow B)) = \Leftrightarrow (v^*(A), v^*(B))$$
DEFINITIONS: Truth Assignment Extension $v^*$ Revisited

Notation

For *binary connectives* (two argument functions) we adopt a convention to write the *symbol of the connective* (name of the 2 argument function) *between its arguments* as we do in a case arithmetic operations.

The *condition (ii)* of the definition of the extension $v^*$ can be hence *written* as follows:

(ii) and for any $A, B \in F$ we put

\[
\begin{align*}
    v^*(\neg A) &= \neg v^*(A); \\
    v^*((A \cap B)) &= v^*(A) \cap v^*(B); \\
    v^*((A \cup B)) &= v^*(A) \cup v^*(B); \\
    v^*((A \Rightarrow B)) &= v^*(A) \Rightarrow v^*(B); \\
    v^*((A \Leftrightarrow B)) &= v^*(A) \Leftrightarrow v^*(B)
\end{align*}
\]
DEFINITIONS: Satisfaction Relation

Definition 4  Let \( v : \text{VAR} \rightarrow \{ T, F \} \)
We say that \( v \) satisfies a formula \( A \in \mathcal{F} \) iff \( v^*(A) = T \)

Notation: \( v \models A \)
We say that \( v \) does not satisfy a formula \( A \in \mathcal{F} \) iff \( v^*(A) \neq T \)

Notation: \( v \not\models A \)
DEFINITIONS: Model, Counter-Model, Classical Tautology

Definition 5
Given a formula $A \in \mathcal{F}$ and $v : \text{VAR} \rightarrow \{T, F\}$
We say that
$v$ is a model for $A$ iff $v \models A$
$v$ is a counter-model for $A$ iff $v \not\models A$

Definition 6
$A$ is a tautology iff for any $v : \text{VAR} \rightarrow \{T, F\}$ we have that $v \models A$

Notation
We write symbolically $\models A$ to denote that $A$ is a classical tautology
DEFINITIONS: Restricted Truth Assignments

**Notation:** for any formula $A$, we denote by $\text{VAR}_A$ a set of all variables that appear in $A$

**Definition 7** Given a formula $A \in \mathcal{F}$, any function

$$v_A : \text{VAR}_A \rightarrow \{T, F\}$$

is called a truth assignment restricted to $A$
DEFINITIONS: Restricted Model, Counter Model

Notation: for any formula $A$, we denote by $VAR_A$ a set of all variables that appear in $A$.

Definition 8 Given a formula $A \in \mathcal{F}$

Any function $w : VAR_A \rightarrow \{T, F\}$ such that $w^*(A) = T$

is called a restricted MODEL for $A$.

Any function $w : VAR_A \rightarrow \{T, F\}$ such that $w^*(A) \neq T$

is called a restricted Counter- MODEL for $A$. 
DEFINITIONS: Models for Sets of Formulas

Consider \( \mathcal{L} = \mathcal{L}\{\neg, \cup, \cap, \Rightarrow\} \) and let \( S \neq \emptyset \) be any non-empty set of formulas of \( \mathcal{L} \), i.e.

\[ S \subseteq \mathcal{F} \]

Definition 9
A truth assignment \( \nu : \text{VAR} \rightarrow \{T, F\} \) is a model for the set \( S \) of formulas if and only if

\[ \nu \models A \quad \text{for all formulas} \quad A \in S \]

We write

\[ \nu \models S \]

to denote that \( \nu \) is a model for the set \( S \) of formulas.
DEFINITIONS: Consistent Sets of Formulas

Definition 10
A non-empty set $G \subseteq \mathcal{F}$ of formulas is called \textbf{consistent} if and only if $G$ has a model, i.e. we have that $G \subseteq \mathcal{F}$ is \textbf{consistent} if and only if there is $v$ such that $v \models G$

Otherwise $G$ is called \textbf{inconsistent}
DEFINITIONS: Independent Statements

Definition 11
A formula $A$ is called **independent** from a non-empty set $G \subseteq \mathcal{F}$ if and only if there are truth assignments $v_1, v_2$ such that

$$v_1 \models G \cup \{A\} \quad \text{and} \quad v_2 \models G \cup \{\neg A\}$$

i.e. we say that a formula $A$ is **independent** if and only if $G \cup \{A\}$ and $G \cup \{\neg A\}$ are **consistent**
Many Valued Extensional Semantics
Definition 11
The extensional semantics $\text{M}$ is defined for a non-empty set of $V$ of logical values of any cardinality
We only assume that the set $V$ of logical values of $\text{M}$ always has a special, distinguished logical value which serves to define a notion of tautology
We denote this distinguished value as $T$
Formal definition of many valued extensional semantics $\text{M}$ for the language $L_{\text{CON}}$ consists of giving definitions of the following main components:
1. Logical Connectives under semantics $\text{M}$
2. Truth Assignment for $\text{M}$
3. Satisfaction Relation, Model, Counter-Model under semantics $\text{M}$
4. Tautology under semantics $\text{M}$
Definition of $\mathbf{M}$ - Extensional Connectives

Given a propositional language $\mathcal{L}_{CON}$ for the set $CON = C_1 \cup C_2$, where $C_1$ is the set of all unary connectives, and $C_2$ is the set of all binary connectives.

Let $V$ be a non-empty set of logical values adopted by the semantics $\mathbf{M}$.

**Definition 12**

Connectives $\triangledown \in C_1$, $\circ \in C_2$ are called $\mathbf{M}$ -extensional iff their semantics $\mathbf{M}$ is defined by respective functions

$$\triangledown : V \rightarrow V \quad \text{and} \quad \circ : V \times V \rightarrow V$$
DEFINITION: Definability of Connectives under a semantics $M$

Given a propositional language $L_{CON}$ and its extensional semantics $M$

We adopt the following definition

Definition 13

A connective $\circ \in CON$ is definable in terms of some connectives $\circ_1, \circ_2, ... \circ_n \in CON$ for $n \geq 1$ under the semantics $M$ if and only if the connective $\circ$ is a certain function composition of functions $\circ_1, \circ_2, ... \circ_n$ as they are defined by the semantics $M$
DEFINITION: M Truth Assignment Extension $v^*$ to $\mathcal{F}$

Definition 14
Given the M truth assignment $v : \text{VAR} \rightarrow V$
We define its M extension $v^*$ to the set $\mathcal{F}$ of all formulas of $\mathcal{L}$ as any function $v^* : \mathcal{F} \rightarrow V$, such that the following conditions are satisfied

(i) for any $a \in \text{VAR}$

$$v^*(a) = v(a);$$

(ii) For any connectives $\nabla \in C_1$, $\circ \in C_2$ and for any formulas $A, B \in \mathcal{F}$ we put

$$v^*(\nabla A) = \nabla v^*(A)$$

$$v^*((A \circ B)) = \circ(v^*(A), v^*(B))$$
**DEFINITION: M Satisfaction, Model, Counter Model, Tautology**

**Definition 15**  Let \( v : \text{VAR} \rightarrow V \)

Let \( T \in V \) be the **distinguished logical value**
We say that
\( v \) **M satisfies** a formula \( A \in \mathcal{F} \) \( (v \models_{M} A) \) iff
\( v^*(A) = T \)

**Definition 16**

Given a formula \( A \in \mathcal{F} \) and \( v : \text{VAR} \rightarrow V \)

Any \( v \) such that \( v \models_{M} A \) is called a **M model** for \( A \)

Any \( v \) such that \( v \not\models_{M} A \) is called a **M counter model** for \( A \)

\( A \) is a **M tautology** \( (\models_{M} A) \) iff \( v \models_{M} A \), for all \( v : \text{VAR} \rightarrow V \)
CHAPTER 3: Some Sample Questions with Solutions
Chapter 3: Question 1

Question 1

Find a restricted model for formula $A$, where

$$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

You can’t use short-hand notation

Show each step of solution

Solution

For any formula $A$, we denote by $\text{VAR}_A$ a set of all variables that appear in $A$

In our case we have $\text{VAR}_A = \{a, b, c\}$

Any function $v_A : \text{VAR}_A \rightarrow \{T, F\}$ is called a truth assignment restricted to $A$
Chapter 3: Question 1

Let $v : \text{VAR} \rightarrow \{T, F\}$ be any truth assignment such that

$$v(a) = v_A(a) = T, \ v(b) = v_A(b) = T, \ v(c) = v_A(c) = F$$

We evaluate the value of the extension $v^*$ of $v$ on the formula $A$ as follows

$$v^*(A) = v^*((\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))))$$

$$= v^*(\neg a) \Rightarrow v^*((\neg b \cup (b \Rightarrow \neg c)))$$

$$= \neg v^*(a) \Rightarrow (v^*(\neg b) \cup v^*((b \Rightarrow \neg c)))$$

$$= \neg v(a) \Rightarrow (\neg v(b) \cup (v(b) \Rightarrow \neg v(c)))$$

$$= \neg v_A(a) \Rightarrow (\neg v_A(b) \cup (v_A(b) \Rightarrow \neg v_A(c)))$$

$$(-T \Rightarrow (-T \cup (T \Rightarrow \neg F))) = F \Rightarrow (F \cup T) = F \Rightarrow T = T,$$ i.e.

$$v_A \models A \quad \text{and} \quad v \models A$$
Chapter 3: Question 2

Question 2
Find a restricted model and a restricted counter-model for \( A \), where
\[
A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))
\]
You can use short-hand notation. Show work.

Solution

Notation: for any formula \( A \), we denote by \( VAR_A \) a set of all variables that appear in \( A \).

In our case we have \( VAR_A = \{a, b, c\} \).

Any function \( v_A: VAR_A \rightarrow \{T, F\} \) is called a truth assignment restricted to \( A \).

We define now \( v_A(a) = T \), \( v_A(b) = T \), \( v_A(c) = F \), in shorthand: \( a = T, b = T, c = F \) and evaluate
\[
(\neg T \Rightarrow (\neg T \cup (T \Rightarrow \neg F))) = F \Rightarrow (F \cup T) = F \Rightarrow T = T, \text{ i.e. }
\]
\[
v_A \models A
\]
Chapter 3: Question 2

Observe that

\((\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))) = T\) when \(a = T\) and \(b, c\) any truth values as by definition of implication we have that \(F \Rightarrow \text{anything} = T\)

Hence \(a = T\) gives us 4 models as we have \(2^2\) possible values on \(b\) and \(c\)
We take as a restricted counter-model: \( a = F, b = T \) and \( c = T \)

**Evaluation:** observe that

\[
(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))) = F \quad \text{if and only if}
\]

\[
\neg a = T \quad \text{and} \quad (\neg b \cup (b \Rightarrow \neg c)) = F \quad \text{if and only if}
\]

\[
a = F, \neg b = F \quad \text{and} \quad (b \Rightarrow \neg c) = F \quad \text{if and only if}
\]

\[
a = F, b = T \quad \text{and} \quad (T \Rightarrow \neg c) = F \quad \text{if and only if}
\]

\[
a = F, b = T \quad \text{and} \quad \neg c = F \quad \text{if and only if}
\]

\[
a = F, b = T \quad \text{and} \quad c = T
\]

The above proves also that \( a = F, b = T \) and \( c = T \) is the only restricted counter-model for \( A \)
Question 3  Justify whether the following statements true or false

S1  There are more than 3 possible restricted counter-models for $A$

S2  There are more than 2 possible restricted models of $A$

Solution

S1 Statement: There are more than 3 possible restricted counter-models for $A$ is false

We have just proved that there is only one possible restricted counter-model for $A$

S2 Statement: There are more than 2 possible restricted models of $A$ is true

There are 7 possible restricted models for $A$

Justification: $2^3 - 1 = 7$
Chapter 3: Question 4

Question 4
1. List 3 models for $A$ from Question 2, i.e. for formula

$$A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$$

that are extensions to the set $VAR$ of all variables of one of the restricted models that you have found in Questions 1,

2. List 2 counter models for $A$ that are extensions of one of the restricted counter models that you have found in the Questions 1, 2
Chapter 3: Question 4

Solution
1. One of the restricted models is, for example a function $v_A : \{a, b, c\} \rightarrow \{T, F\}$ such that $v_A(a) = T$, $v_A(b) = T$, $v_A(c) = F$

We extend $v_A$ to the set of all propositional variables $VAR$ to obtain a (non restricted) models as follows
Chapter 3: Question 4

Model $w_1$ is a function

\[ w_1 : VAR \rightarrow \{T, F\} \quad \text{such that} \]

\[ w_1(a) = v_A(a) = T, \quad w_1(b) = v_A(b) = T, \]

\[ w_1(c) = v_A(c) = F, \quad \text{and} \quad w_1(x) = T, \quad \text{for all} \]

\[ x \in VAR - \{a, b, c\} \]

Model $w_2$ is defined by a formula

\[ w_2(a) = v_A(a) = T, \quad w_2(b) = v_A(b) = T, \]

\[ w_2(c) = v_A(c) = F, \quad \text{and} \quad w_2(x) = F, \quad \text{for all} \]

\[ x \in VAR - \{a, b, c\} \]
Chapter 3: Question 4

Model $w_3$ is defined by a formula

$$w_3(a) = v_A(a) = T, \ w_3(b) = v_A(b) = T, \ w_3(c) = v(c) = F, \ w_3(d) = F \ \text{and} \ \ w_3(x) = T \ \text{for all} \ x \in VAR - \{a, b, c, d\}$$

There is as many of such models, as extensions of $v_A$ to the set $VAR$, i.e. as many as real numbers.
Chapter 3: Question 4

2. A counter-model for a formula $A = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)))$ is, by definition any function

$$v : \text{VAR} \rightarrow \{T, F\}$$

such that $v^*(A) = F$

A restricted counter-model for the formula $A$, the only one, as already proved in is a function

$$v_A : \{a, b\} \rightarrow \{T, F\}$$

such that such that

$$v_A(a) = F, \quad v_A(b) = T, \quad v_A(c) = T$$
We extend $v_A$ to the set of all propositional variables $VAR$ to obtain (non restricted) some counter-models.

Here are two of such extensions

**Counter-model $w_1$:**

$w_1(a) = v_A(a) = F$,  
$w_1(b) = v_A(b) = T$,

$w_1(c) = v(c) = T$, and $w_1(x) = F$, for all $x \in VAR - \{a, b, c\}$

**Counter-model $w_2$:**

$w_2(a) = v_A(a) = T$,  
$w_2(b) = v_A(b) = T$,

$w_2(c) = v(c) = T$, and $w_2(x) = T$ for all $x \in VAR - \{a, b, c\}$

There is as many of such counter-models, as extensions of $v_A$ to the set $VAR$, i.e. as many as real numbers
Definition
A truth assignment $v$ is a model for a set $G \subseteq F$ of formulas of a given language $L = L\{\neg, \Rightarrow, \cup, \cap\}$ if and only if
$$v \models B \text{ for all } B \in G$$
We denote it by $v \models G$
Observe that the set $G \subseteq F$ can be finite or infinite
Chapter 3: Consistent Sets of Formulas

Definition
A set \( G \subseteq \mathcal{F} \) of formulas is called consistent if and only if \( G \) has a model, i.e. we have that

\[
G \subseteq \mathcal{F} \text{ is consistent if and only if there is } v \text{ such that } v \models G
\]

Otherwise \( G \) is called inconsistent.
Definition
A formula $A$ is called independent from a set $G \subseteq \mathcal{F}$ if and only if there are truth assignments $v_1, v_2$ such that

$$v_1 \models G \cup \{A\} \quad \text{and} \quad v_2 \models G \cup \{\neg A\}$$

i.e. we say that a formula $A$ is independent if and only if

$$G \cup \{A\} \quad \text{and} \quad G \cup \{\neg A\} \quad \text{are consistent}$$
Chapter 3: Question 5

Question 5
Given a set \( G = \{(a \cap b) \Rightarrow b, \ (a \cup b), \neg a\} \)
Show that \( G \) is consistent

Solution
We have to find \( v : \text{VAR} \rightarrow \{T, F\} \) such that \( v \models G \)
It means that we need to find a \( v \) such that

\[
\begin{align*}
v^*((a \cap b) \Rightarrow b) &= T, \quad v^*(a \cup b) = T, \quad v^*(\neg a) = T
\end{align*}
\]

We write it in the shorthand notation

\[
\begin{align*}
(a \cap b) \Rightarrow b &= T, \quad a \cup b = T, \quad \neg a = T
\end{align*}
\]

We have to find out of it is possible
Chapter 3: Question 5

1. Observe that $\models ((a \land b) \Rightarrow b)$, hence we have that $\forall^*(a \land b) \Rightarrow b = T$ for any $\forall$

2. Case $\neg a = T$ holds if and only if $a = F$

3. Case $(a \lor b) = T$ holds if and only if $(T \lor b) = T$ as $a = F$, and this holds if and only if $b = T$

This proves that for any $\forall : \text{VAR} \rightarrow \{T, F\}$ such that $\forall(a) = F, \ \forall(b) = T$, is a model for $G$ and so, by definition, that $G$ is consistent

Moreover, we have proved that it is the only (restricted) model for $G$
Chapter 3: Question 6

Question 6
Show that a formula \( A = (\neg a \cap b) \) is not independent of
\[ G = \{((a \cap b) \Rightarrow b), (a \cup b), \neg a\} \]

Solution
We have to show that \textbf{it is impossible} to construct \( v_1, v_2 \) such that
\[ v_1 \models G \cup \{A\} \quad \text{and} \quad v_2 \models G \cup \{\neg A\} \]

Observe that we have just proved that any \( v \) such that \( v(a) = F, \) and \( v(b) = T \) is the only model restricted to the set of variables \( \{a, b\} \) for \( G \) so we have to check now if it is possible that for that formula \( A = (\neg a \cap b), \) \( v \models A \) and \( v \models \neg A \)
Chapter 3: Question 6

We have to evaluate $v^*(A)$ and $v^*(\neg A)$ for $v(a) = F$, and $v(b) = T$
$v(A) = v^*((\neg a \land b)) = \neg v(a) \land v(b) = \neg F \land T = T \land T = T$
and so $v \models A$
$v(\neg A) = \neg v^*(A) = \neg T = F$
and so $v \not\models \neg A$

This ends the proof that $A$ is not independent of $G$
Question 7
Find an infinite number of formulas that are independent of

\[ G = \{((a \cap b) \Rightarrow b), (a \cup b), \neg a\} \]

This my solution - there are many others, but this one seemed to me to be the simplest

Solution
We just proved that any \( v \) such that \( v(a) = F, \ v(b) = T \) is the only model restricted to the set of variables \( \{a, b\} \) and so all other possible models for \( G \) must be extensions of \( v \)
Chapter 3: Question 7

We define a countably infinite set of formulas (and their negations) and corresponding extensions of $\nu$ (restricted to to the set of variables $\{a, b\}$) such that $\nu \models G$ as follows.

Observe that all extensions of $\nu$ restricted to to the set of variables $\{a, b\}$ have as domain the infinitely countable set

$$VAR - \{a, b\} = \{a_1, a_2, \ldots, a_n, \ldots\}$$

We take as a set of formulas (to be proved to be independent) the set of atomic formulas

$$\mathcal{F}_0 = VAR - \{a, b\} = \{a_1, a_2, \ldots, a_n, \ldots\}$$
proof of independence of any formula of $F_0$

Let $c \in F_0$

We define truth assignments $v_1, v_2 : \text{VAR} \rightarrow \{T, F\}$ such that

$v_1 \models G \cup \{c\}$ and $v_2 \models G \cup \{\neg c\}$

as follows

$v_1(a) = v(a) = F$, $v_1(b) = v(b) = T$ and $v_1(c) = T$
for all $c \in F_0$

$v_2(a) = v(a) = F$, $v_2(b) = v(b) = T$ and $v_2(c) = F$
for all $c \in F_0$
CHAPTER 3
Some Extensional Many Valued Semantics
Chapter 3: Question 8

Question 8
We define a 4 valued $H_4$ logic semantics as follows

The language is $\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$

The logical connectives $\neg, \Rightarrow, \cup, \cap$ of $H_4$ are operations in the set $\{F, \bot_1, \bot_2, T\}$, where $\{F < \bot_1 < \bot_2 < T\}$ and are defined as follows

Conjunction $\cap$ is a function

$\cap : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \rightarrow \{F, \bot_1, \bot_2, T\}$,

such that for any $x, y \in \{F, \bot_1, \bot_2, T\}$

$$x \cap y = \min\{x, y\}$$
Disjunction $\cup$ is a function
\[ \cup : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \rightarrow \{F, \bot_1, \bot_2, T\}, \]
such that for any \( x, y \in \{F, \bot_1, \bot_2, T\} \)
\[ x \cup y = \max\{x, y\} \]

Implication $\Rightarrow$ is a function
\[ \Rightarrow : \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \rightarrow \{F, \bot_1, \bot_2, T\}, \]
such that for any \( x, y \in \{F, \bot_1, \bot_2, T\} \),
\[ x \Rightarrow y = \begin{cases} T & \text{if } x \leq y \\ y & \text{otherwise} \end{cases} \]

Negation: for any \( x, y \in \{F, \bot_1, \bot_2, T\} \)
\[ \neg x = x \Rightarrow F \]
Chapter 3: Question 8

Part 1  Write Truth Tables for IMPLICATION and NEGATION in $\mathbf{H}_4$

Solution

$H_4$ Implication

<table>
<thead>
<tr>
<th>$\Rightarrow$</th>
<th>F</th>
<th>$\bot_1$</th>
<th>$\bot_2$</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>$\bot_1$</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>$\bot_2$</td>
<td>F</td>
<td>$\bot_1$</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>$\bot_1$</td>
<td>$\bot_2$</td>
<td>T</td>
</tr>
</tbody>
</table>

$H_4$ Negation

<table>
<thead>
<tr>
<th>$\neg$</th>
<th>F</th>
<th>$\bot_1$</th>
<th>$\bot_2$</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
Chapter 3: Question 7

Part 2 Verify whether

\[ \models_{H_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b)) \]

Solution

Take any \( v \) such that
\[ v(a) = \bot_1 \quad v(b) = \bot_2 \]

Evaluate
\[ v \ast ((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = (\bot_1 \Rightarrow \bot_2) \Rightarrow (\neg \bot_1 \cup \bot_2) = T \Rightarrow (F \cup \bot_2) = T \Rightarrow \bot_2 = \bot_2 \]

This proves that our \( v \) is a counter-model and hence

\[ \not\models_{H_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b)) \]
Chapter 3: Question 9

Question 9
Show that (can’t use TTables!)

\[ \vdash ((\neg a \cup b) \Rightarrow (((c \cap d) \Rightarrow \neg d) \Rightarrow (\neg a \cup b))) \]

Solution
Denote \( A = (\neg a \cup b) \), and \( B = ((c \cap d) \Rightarrow \neg d) \)

Our formula becomes a substitution of a basic tautology

\[ (A \Rightarrow (B \Rightarrow A)) \]

and hence is a tautology
Chapter 3: Challenge Exercise

1. Define your own propositional language $L_{CON}$ that contains also different connectives that the standard connectives $\neg$, $\cup$, $\cap$, $\Rightarrow$

Your language $L_{CON}$ does not need to include all (if any!) of the standard connectives $\neg$, $\cup$, $\cap$, $\Rightarrow$

2. Describe intuitive meaning of the new connectives of your language

3. Give some motivation for your own semantic

4. Define formally your own extensional semantics $M$ for your language $L_{CON}$ - it means

write carefully all Steps 1-4 of the definition of your $M$
Chapter 3: Question 10

Question 10

Definition

Let $S_3$ be a 3-valued semantics for $L_{\neg, \cup, \Rightarrow}$ defined as follows:

$V = \{F, U, T\}$ is the set of logical values with the distinguished value $T$

$x \Rightarrow y = \neg x \cup y$ for any $x, y \in \{F, U, T\}$

$\neg F = T, \quad \neg U = F, \quad \neg T = U$

and

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<tr>
<th>$\cup$</th>
<th>F</th>
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Question 10

Part 1

Consider the following classical tautologies:

\[ A_1 = (a \cup \neg a), \quad A_2 = (a \Rightarrow (b \Rightarrow a)) \]

Find \( S_3 \) counter-models for \( A_1, A_2 \), if exist

You can’t use shorthand notation

Solution

Any \( v \) such that \( v(a) = v(b) = U \) is a counter-model for both \( A_1 \) and \( A_2 \), as

\[
\begin{align*}
\neg v^*(a \cup \neg a) &= v^*(a) \cup \neg v^*(b) = U \cup \neg U = U \cup F = U \neq T \\
v^*(a \Rightarrow (b \Rightarrow a)) &= v^*(a) \Rightarrow (v^*(b) \Rightarrow v^*(a)) = U \Rightarrow (U \Rightarrow U) = U \Rightarrow U = \neg U \cup U = F \cup U = U \neq T
\end{align*}
\]
Question 10

Part 2
Consider the following classical tautologies:

\[ A_1 = (a \cup \neg a), \quad A_2 = (a \Rightarrow (b \Rightarrow a)) \]

Define your own 2-valued semantics \( S_2 \) for \( \mathcal{L} \), such that none of \( A_1, A_2 \) is a \( S_2 \) tautology

Verify your results. You can use shorthand notation.

Solution
This is not the only solution, but it is the simplest and most obvious I could think of! Here it is.

We define \( S_2 \) connectives as follows:
\[ \neg x = F, \quad x \Rightarrow y = F, \quad x \cup y = F \] for all \( x, y \in \{F, T\} \)

Obviously, for any \( v \),
\[ v^*(a \cup \neg a) = F \] and \[ v^*(a \Rightarrow (b \Rightarrow a)) = F \]
Chapter 3: Question 11

Question 11

Prove using proper classical logical equivalences (list them at each step) that for any formulas $A, B$ of language $\mathcal{L}_{\{\neg, \cup, \Rightarrow\}}$

$$\neg (A \iff B) \equiv ((A \cap \neg B) \cup (\neg A \cap B))$$

Solution

$$\neg (A \iff B) \equiv^{\text{def}} \neg ((A \Rightarrow B) \cap (B \Rightarrow A))$$
$$\equiv^{\text{deMorgan}} (\neg (A \Rightarrow B) \cup \neg (B \Rightarrow A))$$
$$\equiv^{\text{negimpl}} ((A \cap \neg B) \cup (B \cap \neg A)) \equiv^{\text{commut}} ((A \cap \neg B) \cup (\neg A \cap B))$$
Question 12

Prove using proper classical logical equivalences (list them at each step) that for any formulas \( A, B \) of language \( \mathcal{L}_{\{\neg, \lor, \Rightarrow\}} \)

\[
((B \cap \neg C) \Rightarrow (\neg A \lor B)) \equiv ((B \Rightarrow C) \lor (A \Rightarrow B))
\]

Solution

\[
((B \cap \neg C) \Rightarrow (\neg A \lor B))
\]

\[
\equiv^{\text{impl}}(\neg(B \cap \neg C) \lor (\neg A \lor B))
\]

\[
\equiv^{\text{deMorgan}}((\neg B \lor \neg C) \lor (\neg A \lor B))
\]

\[
\equiv^{\text{dneg}}((\neg B \lor C) \lor (\neg A \lor B)) \equiv^{\text{impl}}((B \Rightarrow C) \lor (A \Rightarrow B))
\]
Question 13

We define Ł connectives for $L_{\{\neg, \cup, \Rightarrow\}}$ as follows:

Ł Negation $\neg$ is a function:

$$\neg : \{T, \bot, F\} \rightarrow \{T, \bot, F\}$$

such that $\neg \bot = \bot$, $\neg T = F$, $\neg F = T$

Ł Conjunction $\cap$ is a function:

$$\cap : \{T, \bot, F\} \times \{T, \bot, F\} \rightarrow \{T, \bot, F\}$$

such that $x \cap y = min\{x, y\}$ for all $x, y \in \{T, \bot, F\}$

Remember that we assumed: $F < \bot < T$
Ł Implication ⇒ is a function:

⇒: \{T, \bot, F\} \times \{T, \bot, F\} \rightarrow \{T, \bot, F\}

such that

\[ x \Rightarrow y = \begin{cases} 
- x \cup y & \text{if } x > y \\
T & \text{otherwise}
\end{cases} \]

Given a formula \(((a \cap b) \Rightarrow \neg b) \in \mathcal{F}\) of \(\mathcal{L}_{\neg, \cup, \Rightarrow}\)

Use the fact that \(v: \text{VAR} \rightarrow \{F, \bot, T\}\) is such that

\(v^*(((a \cap b) \Rightarrow \neg b)) = \bot\) under Ł semantics to evaluate all possible \(v^*(((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b))\)

You can use shorthand notation
Solution
The formula \((a \cap b) \Rightarrow \neg b) = \bot\) in Ł connectives semantics in two cases written is the shorthand notation as

- **C1** \((a \cap b) = \bot\) and \(\neg b = F\)
- **C2** \((a \cap b) = T\) and \(\neg b = \bot\).

Consider case **C1**
\[\neg b = F, \text{ so } v(b) = T, \text{ and hence } (a \cap T) = v(a) \cap T = \bot\]
if and only if \(v(a) = \bot\)
It means that \(v^*(((a \cap b) \Rightarrow \neg b)) = \bot\) for any \(v\), is such that \(v(a) = \bot\) and \(v(b) = T\)
Question 13 Solution

We now evaluate (in shorthand notation)

\[ v^*(((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b)) \]

\[ = (((T \Rightarrow \neg \bot) \Rightarrow (\bot \Rightarrow \neg T)) \cup (\bot \Rightarrow T)) = ((\bot \Rightarrow \bot) \cup T) = T \]

Consider now **Case C2**

\[ \neg b = \bot, \text{ i.e. } b = \bot \]

...and hence \((a \cap \bot) = T\) what is impossible, hence \(v\) from the **Case C1** is the only one
Question 14

Use the **Definability of Conjunction** in terms of disjunction and negation **Equivalence**

\[(A \cap B) \equiv \neg(\neg A \cup \neg B)\]

to transform a formula

\[A = \neg(\neg(\neg a \cap \neg b) \cap a)\]

of the language \(L_{\{\cap, \neg}\}\) into a logically equivalent formula \(B\)

of the language \(L_{\{\cup, \neg}\}\)
Question 14

Solution

\[ \neg(\neg(a \cap \neg b) \cap a) \equiv \neg(\neg(a \cap \neg b) \cup \neg a) \]

\[ \equiv ((\neg a \cap \neg b) \cup \neg a) \equiv (\neg(\neg a \cup \neg b) \cup \neg a) \]

\[ \equiv \neg(a \cup b) \cup \neg a \]

The formula \( B \) of \( L_{\{\cup, \neg}\} \) equivalent to \( A \) is

\[ B = (\neg(a \cup b) \cup \neg a) \]
Equivalence of Languages Definition

Definition

Given two languages: $L_1 = L_{\text{CON}_1}$ and $L_2 = L_{\text{CON}_2}$, for $\text{CON}_1 \neq \text{CON}_2$

We say that they are logically equivalent, i.e.

$$L_1 \equiv L_2$$

if and only if the following conditions $\text{C1}, \text{C2}$ hold.

**C1:** for any formula $A$ of $L_1$, there is a formula $B$ of $L_2$, such that $A \equiv B$

**C2:** for any formula $C$ of $L_2$, there is a formula $D$ of $L_1$, such that $C \equiv D$
Question 14

Prove the logical equivalence of the languages

\[ \mathcal{L}_{\{\neg, \cup\}} \equiv \mathcal{L}_{\{\neg, \Rightarrow\}} \]

Solution

We need two definability equivalences:

implication in terms of disjunction and negation

\[ (A \Rightarrow B) \equiv (\neg A \cup B) \]

and disjunction in terms of implication and negation,

\[ (A \cup B) \equiv (\neg A \Rightarrow B) \]

and the Substitution Theorem
Question 15

Prove the logical equivalence of the languages

\[ \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}} \equiv \mathcal{L}_{\{\neg, \cap, \cup\}} \]

Solution

We need only the **definability of implication** in terms of disjunction and negation equivalence

\[ (A \Rightarrow B) \equiv (\neg A \cup B) \]

as the **Substitution Theorem** for any formula \( A \) of \( \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}} \)

**there is** a formula \( B \) of \( \mathcal{L}_{\{\neg, \cap, \cup\}} \) such that \( A \equiv B \) and the condition **C1** holds

**Observe** that any formula \( A \) of language \( \mathcal{L}_{\{\neg, \cap, \cup\}} \) is also a

formula of the language \( \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}} \) and of course \( A \equiv A \) so

the condition **C2** also holds
Question 16

Prove that

\[ L_{\neg, \cap} \equiv L_{\neg, \Rightarrow} \]

Solution

The equivalence of languages holds due to the following two definability of connectives equivalences, respectively

\[(A \cap B) \equiv \neg(A \Rightarrow \neg B), \quad (A \Rightarrow B) \equiv \neg(A \cap \neg B)\]

and Substitution Theorem
Question 17

Prove that in classical semantics

\[ \mathcal{L}_{\neg, \Rightarrow} \equiv \mathcal{L}_{\neg, \Rightarrow, \cup} \]

Solution

OBSERVE that the condition \textbf{C1} holds because any formula of \( \mathcal{L}_{\neg, \Rightarrow} \) is also a formula of \( \mathcal{L}_{\neg, \Rightarrow, \cup} \).

Condition \textbf{C2} holds due to the following definability of connectives equivalence

\[(A \cup B) \equiv (\neg A \Rightarrow B)\]

and \textbf{Substitution Theorem}
Question 18

Prove that the equivalence defining $\cup$ in terms of negation and implication in classical logic does not hold under $\mathcal{L}$ semantics, i.e. that

$$(A \cup B) \not\equiv_{\mathcal{L}} (\neg A \Rightarrow B)$$

but nevertheless

$${\mathcal{L}\{\neg,\Rightarrow\}} \equiv_{\mathcal{L}} {\mathcal{L}\{\neg,\Rightarrow,\cup\}}$$
Question 18

Solution

We prove

\[ L_{\neg, \rightarrow} \equiv_L L_{\neg, \rightarrow, \cup} \]

as follows

Condition C2 holds because the definability of connectives equivalence

\[(A \cup B) \equiv_L ((A \rightarrow B) \rightarrow B)\]

Check it by verification as an exercise

C1 holds because any formula of \( L_{\neg, \rightarrow} \) is a formula of \( L_{\neg, \rightarrow, \cup} \)

Observe that the equivalence \( (A \cup B) \equiv (A \rightarrow B) \rightarrow B \) provides also an alternative proof of C2 in classical case