

# CSE581

## Computer Science Fundamentals: Theory

Professor Anita Wasilewska

## P1 LOGIC: LECTURE 3c

## Chapter 3

# Propositional Semantics: Classical and Many Valued

## Extensional Semantics **M**

## Extensional Semantics **M** - Introduction

Given a propositional language  $\mathcal{L}_{CON}$ , the symbols for its **connectives** always have some intuitive **meaning**

A formal **definition** of the **meaning** of these **symbols** is called a **semantics** for the language  $\mathcal{L}_{CON}$

A given language  $\mathcal{L}_{CON}$  can have different **semantics** but we always **define** them in order to single out **special formulas** of the language, called **tautologies**

**Tautologies** are formulas of the language that are **always true** under a given **semantics**

## Extensional Semantics **M** Introduction

We have already introduced the **intuitive** and **formal** notions of a classical **semantics**, discussed its **motivation** and underlying **assumptions**

The **classical semantics** assumption is that it considers only **two** logical values. The other one is that all classical propositional **connectives** are **extensional**

We have also observed that in everyday language there are expressions such as "I believe that", "it is possible that", "certainly", etc .... and that they are represented by some propositional **connectives** which are **not extensional**

## Extensional Semantics **M** Introduction

**Non-extensional** connectives **do not** play any role in **mathematics** and so **are not** discussed in **classical logic** and will be studied separately

The **extensional connectives** are defined **intuitively** as such that the **logical value** of the formulas form by means of these **connectives** and certain given formulas **depends only** on the **logical value(s)** of the given formulas

## Extensional Connectives Definition

We **adopt** a following **formal** definition of **extensional** connectives for a propositional language  $\mathcal{L}_{CON}$

### Definition

Let  $\mathcal{L}_{CON}$  be such that  $CON = C_1 \cup C_2$ , where  $C_1, C_2$  are the sets of **unary** and **binary** connectives, respectively

Let  $LV$  be a non-empty set of **logical values**

A connective  $\nabla \in C_1$  or  $\circ \in C_2$  is called **extensional** if it is defined by a respective function

$$\nabla : LV \longrightarrow LV \quad \text{or} \quad \circ : LV \times LV \longrightarrow LV$$

## Extensional Semantics **M** Introduction

A semantics **M** for a language  $\mathcal{L}_{CON}$  is called **extensional** provided all connectives in **CON** are **extensional** and its notion of **tautology** is defined in terms of the connectives and their logical values

A semantics with a set of **m** logical values is called a **m-valued extensional**

The **classical** semantics is a special case of a **2-valued extensional** semantics

Classical **semantics** **defines** classical **logic** with its set of classical propositional **tautologies**

Many of logics are defined by various **extensional semantics** with sets of logical values **LV** with more than **2 elements**



## Extensional Semantics **M** Introduction

The languages of many important **logics** like **modal**, **multi-modal**, **knowledge**, **believe**, **temporal**, contain **connectives** that are **not extensional** because they are defined by **non-extensional** semantics

The **intuitionistic logic** is based on the **same** language as the **classical** one and its **Kripke Models** semantics is **not extensional**

Defining a **semantics** for a given **language** means **more** than defining **connectives**

The ultimate **goal** of any semantics is to **define** the notion of its own **tautology**

## Extensional Semantics **M** Introduction

In order to **define** which formulas of a given

$$\mathcal{L}_{CON}$$

we want to be **tautologies** under a given **semantics M** we **assume** that the set **LV** of logical values of **M** always has a **distinguished** logical value, often denoted by **T** for "absolute" **truth**

We also can **distinguish**, and often we do, another special value **F** representing "absolute" **falsehood**

We will use these symbols **T**, **F** for "absolute" **truth** and **falsehood**

We may also use other symbols like **1**, **0** or **others**

## Extensional Semantics **M** Introduction

The "absolute" **truth** value **T** serves to **define** a notion of a **tautology** (as a formula always "true")

**Extensional semantics** share not only the similar **pattern** of **defining** their (extensional) **connectives**, but also the method of **defining** the notion of a **tautology**

We hence **define** a general notion of an **extensional semantics** as sequence of **steps** leading to the definition of a **tautology**

## Extensional Semantics **M** Introduction

Here are the **steps** leading to the definition of a **tautology**

**Step 1** We **define** all extensional **connectives** of **M**

**Step 2** We **define** main component of the definition of a **tautology**, namely a **function** **v** that assigns to any formula  $A \in \mathcal{F}$  its logical **value** from **LV**

The function **v** is often called a **truth assignment** and we will use this name

## Extensional Semantics **M** Introduction

**Step 3** Given a truth assignment  $v$  and a formula  $A \in \mathcal{F}$ , we **define** what does it mean that

$v$  **satisfies**  $A$

i.e. we define a notion saying that  $v$  is a **model** for  $A$  under semantics **M**

**Step 4** We **define** a notion of tautology as follows

$A$  is a **tautology** under semantics **M** if and only if **all** truth assignments  $v$  **satisfy**  $A$

i.e. that **all** truth assignments  $v$  are **models** for  $A$

## Extensional Semantics **M** Introduction

We use a notion of a **model** because it is an important, if not the **most important** notion of modern **logic**

The notion of a **model** is usually **defined** in terms of the notion of **satisfaction**

In **classical** propositional logic these notions are the **same** and the **use** of expressions

"**v** **satisfies** **A**" and "**v** is a **model** for **A**"  
is **interchangeable**

This also **is true** for of any propositional **extensional semantics** and in particular it holds for **m-valued** semantics discussed later in this chapter

## Extensional Semantics **M** Introduction

The notions of **satisfaction** and **model** are not interchangeable for **predicate** languages semantics

We already discussed **intuitively** the notion of **model** and **satisfaction** for **predicate** language in chapter 2

We will define them in **full formality** in chapter 8

The use of the notion of a **model** also allows us to adopt and discuss the **standard** predicate logic **definitions** of **consistency** and **independence** for **propositional** case

## Extensional Semantics **M** Formal Definition

### Definition

Any formal definition of an **extensional semantics** **M** for a given language  $\mathcal{L}_{CON}$  consists of **specifying** the following steps **defining** its main components

**Step 1** We define a set  $LV$  of logical values, its distinguished value  $T$ , and define all connectives of  $\mathcal{L}_{CON}$  to be **extensional**

**Step 2** We define notion of a **truth assignment** and its **extension**

**Step 3** We define notions of **satisfaction**, **model**, **counter model**

**Step 4** We define notion of a **tautology** under the semantics **M**



## Extensional Semantics **M** Formal Definition

What **differs** one semantics from the other is the **choice** of the set **LV** of logical values and **definition** of the connectives of  **$\mathcal{L}_{CON}$** , that are defined in the first step below

**Step 1** We adopt a following **formal** definition of **extensional** connectives of  **$\mathcal{L}_{CON}$**

### Definition

Let  **$\mathcal{L}_{CON}$**  be such that  **$CON = C_1 \cup C_2$** , where  **$C_1, C_2$**  are the sets of **unary** and **binary** connectives, respectively

Let **LV** be a non-empty set of **logical values**

A connective  **$\nabla \in C_1$**  or  **$\circ \in C_2$**  is called **extensional** if it is defined by a respective function

$$\nabla : LV \longrightarrow LV \quad \text{or} \quad \circ : LV \times LV \longrightarrow LV$$

## M Truth Assignment Formal Definition

**Step 2** We define a function called **truth assignment** and its **extension** in terms of the **propositional connectives** as defined in the **Step 1**

### Definition

Let **LV** be the set of logical values of **M** and **VAR** the set of propositional variables of the language  $\mathcal{L}_{CON}$

Any function

$$v : VAR \longrightarrow LV$$

is called a **truth assignment** under semantics **M**

We call it for short a **M truth assignment**

We use the term **M** truth assignment and **M** truth extension to stress that it is defined **relatively** to a given semantics **M**

## M Truth Extension Formal Definition

### Definition

Given a **M** truth assignment  $v : VAR \rightarrow LV$

We define its **extension**  $v^*$  to the set  $\mathcal{F}$  of all formulas of  $\mathcal{L}_{CON}$  as any function

$$v^* : \mathcal{F} \rightarrow LV$$

such that the following conditions are satisfied.

(i) for any  $a \in VAR$ ,

$$v^*(a) = v(a);$$

(ii) For any connectives  $\nabla \in C_1$ ,  $\circ \in C_2$ , and for any formulas  $A, B \in \mathcal{F}$ ,

$$v^*(\nabla A) = \nabla v^*(A) \quad \text{and} \quad v^*((A \circ B)) = \circ(v^*(A), v^*(B))$$

We call the  $v^*$  the **M truth extension**

## M Truth Extension Formal Definition

### Remark

The **symbols** on the **left-hand side** of the equations

$$v^*(\nabla A) = \nabla v^*(A) \quad \text{and} \quad v^*((A \circ B)) = \circ(v^*(A), v^*(B))$$

**represent** connectives in their **natural language** meaning and the symbols on the **right-hand side** represent connectives in their **semantical meaning** as defined in the **Step1**

## M Truth Extension Formal Definition

We use names " **M truth assignment**" and " **M truth extension**" to stress that we define them for the set of logical values of the semantics **M**

### Notation Remark

For any function  $g$ , we use a symbol  $g^*$  to denote its **extension** to a **larger domain**

Mathematician often use the same symbol  $g$  for both a function  $g$  and its extension  $g^*$

## Satisfaction and Model

**Step 3** The notions of **satisfaction** and **model** are **interchangeable** in **M** semantics and we define them as follows.

### Definition

Given an **M** truth assignment  $v : VAR \rightarrow LV$  and its **M** truth extension  $v^*$  Let  $T \in LV$  be the distinguished logical truth value

We say that the truth assignment  $v$  **M satisfies** a formula  $A$  if and only if  $v^*(A) = T$

We write symbolically

$$v \models_M A$$

Any truth assignment  $v$ , such that  $v \models_M A$  is called an **M model** for the formula  $A$

## Counter Model

### Definition

Given an **M** truth assignment  $v : VAR \rightarrow LV$  and its **M** truth extension  $v^*$ . Let  $T \in LV$  be the distinguished logical truth value

We say that the truth assignment  $v$  **M does not satisfy** a formula  $A$  if and only if  $v^*(A) \neq T$

We write symbolically

$$v \not\models_M A$$

Any truth assignment  $v$ , such that  $v \not\models_M A$  is called an **M counter model** for the formula  $A$

## **M** Tautology

**Step 4** We define the notion of **M tautology** as follows

### Definition

A formula **A** is an **M tautology** if and only if

$v \models_{\mathbf{M}} A$ , for all truth assignments  $v : VAR \rightarrow LV$

We denote it as

$$\models_{\mathbf{M}} A$$

We also say that

**A** is an **M tautology** if and only if all truth assignments  $v : VAR \rightarrow LV$  are **M models** for **A**



## M Tautology

Observe that directly from definition of the **M model** we get the following equivalent form of the definition of **tautology**

### Definition

A formula **A** is an **M tautology** if and only if

$v^*(A) = T$ , for all truth assignments  $v : VAR \longrightarrow LV$

We denote by **MT** the set of **all tautologies** under the semantic **M**, i.e.

$$MT = \{A \in \mathcal{F} : \models_M A\}$$

## M Tautology

Obviously, when we **develop a logic** by defining its **semantics** we want the semantics to be such that the **logic** has a **non empty** set of its tautologies

We **express** it in a form of the following definition

### Definition

Given a language  $\mathcal{L}_{CON}$  and its extensional semantics **M**

The semantics **M** is **well defined** if and only if its set **MT** of all tautologies is **non empty**, i.e. when

$$MT \neq \emptyset$$

## Extensional Semantics **M**

As the **next steps** we use the **definitions** established here to define and discuss in details the following **particular** cases of the extensional semantics **M**

Many valued **semantics** have their beginning in the work of Łukasiewicz (1920)

He was the first to define a **3-valued** extensional semantics for a language  $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$  of classical logic, and called it a **3-valued** logic, for short

## Extensional Semantics **M**

The other **logics** defined by various **extensional semantics** followed and we will discuss some of them

In particular we present **Heyting's 3-valued semantics** as an introduction to the discussion of **first** ever semantics for the **intuitionistic logic** and some **modal logics**

## Challenge Exercise

1. **Define** your own propositional language  $\mathcal{L}_{CON}$  that contains also **different connectives** that the standard connectives  $\neg, \cup, \cap, \Rightarrow$

Your language  $\mathcal{L}_{CON}$  **does not need** to include all (if any!) of the standard connectives  $\neg, \cup, \cap, \Rightarrow$

2. **Describe** intuitive meaning of the new connectives of your language

3. **Give** some **motivation** for **your own** semantic **M**

4. **Define** formally **your own** extensional semantics **M** for your language  $\mathcal{L}_{CON}$

Write carefully all **Steps 1- 4** of the definition of your **M**