# CSE581 Computer Science Fundamentals: Theory

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## P1 LOGIC: LECTURE 3a

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## Chapter 3 Propositional Semantics: Classical and Many Valued

**Classical Semantics** 

### Semantics- General Principles

Given a propositional language  $\mathcal{L} = \mathcal{L}_{CON}$ Symbols for connectives of  $\mathcal{L}$  always have some intuitive meaning

**Semantics** provides a formal definition of the meaning of these symbols

It provides a **method** of **defining formally** a notion of **tautology** under a given **semantics** 

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## **Extensional Connectives**

In **Chapter 2** we described the intuitive classical propositional **semantics** and introduced the following notion of extensional connectives

**Extensional connectives** are the propositional connectives that have the following property:

the logical value of the formulas form by means of these connectives and certain given formulas depends only on the logical value(s) of the given formulas

We also assumed that

All classical propositional connectives

```
\neg, \cup, \cap, \Rightarrow, \Leftrightarrow, \uparrow, \downarrow
```

are extensional

### Non-Extensional Connectives

We have also observed the following

### Remark

In everyday language there are expressions such as "I believe that", "it is possible that", " certainly", etc.... They are represented by some **propositional connectives** which **are not extensional** 

**Non- extensional** connectives do not play any role in **mathematics** and so are not discussed in **classical logic** and will be studied separately

### **Definition of Extensional Connectives**

Given a propositional language  $\mathcal{L}_{CON}$  for the set  $CON = C_1 \cup C_2$ , where  $C_1$  is the set of all unary connectives, and  $C_2$  is the set of all binary connectives Let V be a non-empty set of **logical values** We adopt now a following formal definition of extensional connectives

## Definition

Connectives  $\nabla \in C_1$ ,  $\circ \in C_2$  are called **extensional** 

if and only if their semantics is defined by respective functions

 $\nabla: V \longrightarrow V$  and  $\circ: V \times V \longrightarrow V$ 

Functional Dependency and Definability of Connectives

In **Chapter 2** we talked about **functional dependency** of connectives and of **definability** of a connective in terms of other connectives

We define these notions formally as follows

Functional Dependency and Definability of Connectives

Given a propositional language  $\mathcal{L}_{CON}$  and an **extensional** semantics for it; i.e a semantics such that all connectives in  $\mathcal{L}$  are extensional

## Definition

Connectives  $\circ \in CON$  and  $\circ_1, \circ_2, ... \circ_n \in CON$  (for  $n \ge 1$ ) are functionally dependent iff  $\circ$  is a certain function composition of functions  $\circ_1, \circ_2, ... \circ_n$ 

## Definition

A connective  $\circ \in CON$  is **definable** in terms of some connectives  $\circ_1, \circ_2, ... \circ_n \in CON$  iff  $\circ \in CON$  and  $\circ_1, \circ_2, ... \circ_n \in CON$  are **functionally dependent** 

**Classical Propositional Semantics Assumptions** 

#### Assumptions

A1: We define our semantics for the language

 $\mathcal{L} = \mathcal{L}_{\{\neg, \ \cup, \ \cap, \ \Rightarrow, \ \Leftrightarrow\}}$ 

A2: Two values: the set of logical values  $V = \{T, F\}$ Logical values T, F denote truth and falsehood, respectively There are other notations, for example 0,1

A3: Extensionality: all connectives of  $\mathcal{L}$  are extensional

**Semantics** for any language  $\mathcal{L}$  for which the assumption **A3** holds is called **extensional semantics** 

### **Propositional Semantics Definition**

Formal definition of a propositional **extensional semantics** for a given language  $\mathcal{L}_{CON}$  consists of providing **definitions** of the following four main components:

- 1. Extensional Connectives
- 2. Truth Assignment
- 3. Satisfaction, Model, Counter-Model
- 4. Tautology

The definition of the **classical semantics** and **extensional semantics** for some non-classical logics considered here will follow **the same pattern** 

## Semantics Definition Step 1

The assumption of **extensionality of connectives** means that unary connectives are **functions** defined on a set  $\{T, F\}$ with values in the set  $\{T, F\}$  and binary connectives are **functions** defined on a set  $\{T, F\} \times \{T, F\}$  with values in the set  $\{T, F\}$ In particular we adopt the following definitions **Negation Definition Negation**  $\neg$  is a **function**:

 $\neg: \ \{T,F\} \longrightarrow \{T,F\},$ 

such that

$$\neg T = F, \ \neg F = T$$

## Notation

When defining connectives as functions we usually write the name of a function (our connective) **between the arguments**, not in front as in function notation, i.e. for example we write  $T \cap T = T$  instead of  $\cap(T, T) = T$ 

## Conjunction Definition Conjunction $\cap$ is a function:

 $\cap: \{T,F\} \times \{T,F\} \longrightarrow \{T,F\},$ 

such that

 $\cap(T,T)=T, \quad \cap(T,F)=F, \quad \cap(F,T)=F, \quad \cap(F,F)=F$ 

We write it as

 $T \cap T = T$ ,  $T \cap F = F$ ,  $F \cap T = F$ ,  $F \cap F = F$ 

### **Disjunction Definition**

**Disjunction**  $\cup$  is a function:

 $\cup: \{T, F\} \times \{T, F\} \longrightarrow \{T, F\}$ 

such that

$$\cup(T,T)=T, \quad \cup(T,F)=T, \quad \cup(F,T)=T, \quad \cup(F,F)=F$$

We write it as

 $T \cup T = T$ ,  $T \cup F = T$ ,  $F \cup T = T$ ,  $F \cup F = F$ 

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### **Implication Definition**

**Implication**  $\Rightarrow$  is a function:

 $\Rightarrow: \{T, F\} \times \{T, F\} \longrightarrow \{T, F\}$ 

such that

$$\Rightarrow (T,T) = T, \quad \Rightarrow (T,F) = F, \quad \Rightarrow (F,T) = T, \quad \Rightarrow (F,F) = T$$

We write it as

 $T \Rightarrow T = T, \quad T \Rightarrow F = F, \quad F \Rightarrow T = T, \quad F \Rightarrow F = T$ 

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### **Equivalence Definition**

**Equivalence**  $\Leftrightarrow$  is a function:

 $\Leftrightarrow: \{T, F\} \times \{T, F\} \longrightarrow \{T, F\}$ 

such that

 $\Leftrightarrow (T,T) = T, \quad \Leftrightarrow (T,F) = F, \quad \Leftrightarrow (F,T) = F, \quad \Leftrightarrow (T,T) = T$ 

We write it as

 $T \Leftrightarrow T = T$ ,  $T \Leftrightarrow F = F$ ,  $F \Leftrightarrow T = F$ ,  $T \Leftrightarrow T = T$ 

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**Classical Connectives Truth Tables** 

We write the functions defining connectives in a form of tables, usually called the classical truth tables

Negation:

$$\neg T = F, \ \neg F = T$$
$$\frac{\neg | T | F}{| F | T}$$

Conjunction:

$$T \cap T = T$$
,  $T \cap F = F$ ,  $F \cap T = F$ ,  $F \cap F = F$ 

∩ T F
 T F F
 F F F

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## **Classical Connectives Truth Tables**

Disjunction:

$$T \cup T = T, \quad T \cup F = T, \quad F \cup T = T, \quad F \cup F = F$$

$$\begin{array}{c|c} \cup & T & F \\ \hline T & T & T \\ \hline F & T & F \end{array}$$

Implication:

$$T \Rightarrow T = T, \ T \Rightarrow F = F, \ F \Rightarrow T = T, \ F \Rightarrow F = T$$

$$\Rightarrow \begin{array}{c} T \\ \hline T \\ \hline T \\ F \\ \hline T \\ \hline \end{array}$$

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**Classical Connectives Truth Tables** 

Equivalence:

$$T \Leftrightarrow T = T, \ T \Leftrightarrow F = F, \ F \Leftrightarrow T = F, \ F \Leftrightarrow F = T$$
$$\Leftrightarrow | \begin{array}{c} \mathsf{T} & \mathsf{F} \\ \hline \mathsf{T} & \mathsf{T} & \mathsf{F} \\ \hline \mathsf{F} & \mathsf{F} & \mathsf{T} \end{array}$$

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This ends the Step1 of the semantics definition

## **Definability of Classical Connectives**

## We adopted the following definition

## Definition

A connective  $\circ \in CON$  is **definable** in terms of some connectives  $\circ_1, \circ_2, ... \circ_n \in CON$  iff  $\circ$  is a **certain function composition** of functions  $\circ_1, \circ_2, ... \circ_n$ 

## Example

Classical implication  $\Rightarrow$  is **definable** in terms of  $\cup$  and  $\neg$  because  $\Rightarrow$  can be defined as a **composition** of functions  $\neg$  and  $\cup$ 

**More precisely**, a function  $h: \{T, F\} \times \{T, F\} \longrightarrow \{T, F\}$ defined by a formula

$$h(x,y) = \cup (\neg x,y)$$

is a **composition of functions**  $\neg$  and  $\cup$  and we **prove** that the implication function  $\Rightarrow$  is equal with h

### Short Review: Equality of Functions

#### Definition

Given two sets A, B and functions f, g such that

 $f: A \longrightarrow B$  and  $g: A \longrightarrow B$ 

We say that the functions f, g are **equal** and write is as f = gif and only if f(x) = g(x) for all elements  $x \in A$ **Example:** Consider functions

 $\Rightarrow: \{T, F\} \times \{T, F\} \longrightarrow \{T, F\} \text{ and } h: \{T, F\} \times \{T, F\} \longrightarrow \{T, F\}$ 

where  $\Rightarrow$  is classical implication and the function h is defined by the formula  $h(x, y) = \cup(\neg x, y)$ We **prove** that  $\Rightarrow = h$  by evaluating that  $\Rightarrow (x, y) = h(x, y) = \cup(\neg x, y)$ , for all  $(x, y) \in \{T, F\} \times \{T, F\}$  Definability of Classical Implication

We re-write formula  $\Rightarrow$  (*x*, *y*) =  $\cup(\neg x, y)$  in our adopted notation as

 $x \Rightarrow y = \neg x \cup y$  for all  $(x, y) \in \{T, F\} \times \{T, F\}$ 

and call it a **formula defining**  $\Rightarrow$  in terms of  $\cup$  and  $\neg$ We verify correctness of the **definition** as follows

$T \Rightarrow T = T$	and	$\neg T \cup T = F \cup T = T$	yes
$T \Rightarrow F = F$	and	$\neg T \cup F = F \cup F = F$	yes
$F \Rightarrow F = T$	and	$\neg F \cup F = T \cup F = T$	yes

 $F \Rightarrow T = T$  and  $\neg F \cup T = T \cup T = T$  yes

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**Definability of Classical Connectives** 

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Exercise 1
Find a formula defining \cap, \Leftrightarrow in terms of \cup and \neg
Exercise 2
Find a formula defining
\Rightarrow, \cup, \Leftrightarrow in terms of \cap and \neg
Exercise 3
Find a formula defining \cap, \cup, \Leftrightarrow in terms of \Rightarrow and \neg
Exercise 4
Find a formula defining \cup in terms of \Rightarrow alone
```

**Two More Classical Connectives** 

Sheffer Alternative Negation

 $\uparrow: \{T, F\} \times \{T, F\} \longrightarrow \{T, F\}$ 

such that

 $T \uparrow T = F$ ,  $T \uparrow F = T$ ,  $F \uparrow T = T$ ,  $F \uparrow F = T$ 

Łukasiewicz Joint Negation J

 $\downarrow: \{T, F\} \times \{T, F\} \longrightarrow \{T, F\}$ 

such that

 $T \downarrow T = F$ ,  $T \downarrow F = F$ ,  $F \downarrow T = F$ ,  $F \downarrow F = T$ 

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**Definability of Classical Connectives** 

## **Exercise 4**

Show that the **Sheffer Alternative Negation**  $\uparrow$  defines all classical connectives  $\neg$ ,  $\Rightarrow$ ,  $\cup$ ,  $\cap$ ,  $\Leftrightarrow$ 

## Exercise 5

Show that **Łukasiewicz Joint Negation**  $\downarrow$  defines all classical connectives  $\neg$ ,  $\Rightarrow$ ,  $\cup$ ,  $\cap$ ,  $\Leftrightarrow$ 

## **Exercise 6**

Show that the two binary connectives:  $\downarrow$  and  $\uparrow$  suffice, each of them separately, to define **all classical connectives**, whether unary or binary

## Semantics: Truth Assignment

## Step 2

We define the next components of the classical propositional **semantics** in terms of the propositional connectives as defined in the **Step 1** and a function called truth assignment **Definition** 

A truth assignment is any function

 $v: VAR \longrightarrow \{T, F\}$ 

**Observe** that the domain of **truth assignment** is the set of propositional variables, i.e. the **truth assignment** is defined **only** for **atomic formulas** 

### Truth Assignment Extension

We now **extend** the truth assignment v to the set of **all** formulas  $\mathcal{F}$  in order define formally the logical value for any formula  $A \in \mathcal{F}$ 

The definition of the **extension** of the variable assignment v to the set  $\mathcal{F}$  follows the same pattern for the all extensional connectives, i.e. for **all extensional semantics** 

Truth Assignment Extension  $v^*$  to  $\mathcal{F}$ 

### Definition

Given the truth assignment

 $v: VAR \longrightarrow \{T, F\}$ 

We define its **extension**  $v^*$  to the set  $\mathcal{F}$  of all formulas of  $\mathcal{L}$  as any function

 $v^*: \mathcal{F} \longrightarrow \{T, F\}$ 

such that the following conditions are satisfied

(i) for any  $a \in VAR$  (atomic formula)

$$v^{*}(a) = v(a);$$

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#### Truth Assignment Extension $v^*$ to $\mathcal{F}$

(ii) and for any  $A, B \in \mathcal{F}$  we put

$$v^*(\neg A) = \neg v^*(A);$$
$$v^*((A \cap B)) = \cap (v^*(A), v^*(B));$$
$$v^*((A \cup B)) = \cup (v^*(A), v^*(B));$$
$$v^*((A \Rightarrow B)) = \Rightarrow (v^*(A), v^*(B));$$
$$v^*((A \Leftrightarrow B)) = \Leftrightarrow (v^*(A), v^*(B));$$

The symbols on the **left-hand side** of the equations represent connectives in their **natural language meaning** and the symbols on the **right-hand side** represent connectives in their **semantical meaning** given by the classical truth tables

## Extension v\* Definition Revisited

## Notation

For binary connectives (two argument functions) we adopt a convention to write the symbol of the connective (name of the 2 argument function) between its arguments as we do in a case arithmetic operations

The **condition (ii)** of the definition of the extension  $v^*$  can be hence written as follows

(ii) and for any  $A, B \in \mathcal{F}$  we put

$$v^{*}(\neg A) = \neg v^{*}(A);$$
  

$$v^{*}((A \cap B)) = v^{*}(A) \cap v^{*}(B);$$
  

$$v^{*}((A \cup B)) = v^{*}(A) \cup v^{*}(B);$$
  

$$v^{*}((A \Rightarrow B)) = v^{*}(A) \Rightarrow v^{*}(B);$$
  

$$v^{*}((A \Leftrightarrow B)) = v^{*}(A) \Leftrightarrow v^{*}(B);$$

We will use this notation for the rest of the book

## Truth Assignment Extension Example

Consider a formula

 $((a \Rightarrow b) \cup \neg a))$ 

and a truth assignment v such that

v(a) = T, v(b) = F

Observe that we did not specify v(x) of any  $x \in VAR - \{a, b\}$ , as these values do not influence the computation of the logical value  $v^*(A)$  of the formula A

We say: "*v* such that" - as we consider its values for the set  $\{a, b\} \subseteq VAR$ 

Nevertheless, the domain of v is the set of all variables VAR and we have to **remember** that.

### Truth Assignment Extension Example

Given a formula A:  $((a \Rightarrow b) \cup \neg a))$  and a truth assignment v such that v(a) = T, v(b) = FWe calculate the logical value of the formula A as follows:  $v^*(A) = v^*(((a \Rightarrow b) \cup \neg a))) = \cup (v^*((a \Rightarrow b), v^*(\neg a)) =$  $\cup (\Rightarrow (v^*(a), v^*(b)), \neg v^*(a))) = \cup (\Rightarrow (v(a), v(b)), \neg v(a))) =$  $\cup (\Rightarrow (T, F), \neg T)) = \cup (F, F) = F$ 

We can also calculate it as follows:

$$v^*(A) = v^*(((a \Rightarrow b) \cup \neg a))) = v^*((a \Rightarrow b)) \cup v^*(\neg a) = (v(a) \Rightarrow v(b)) \cup \neg v(a) = (T \Rightarrow F) \cup \neg T = F \cup F = F$$

We write it in a short-hand notation as

$$(T \Longrightarrow F) \cup \neg T = F \cup F = F$$

On **tests** I will specify when you can use the **short-hand notation**.

#### Semantics: Satisfaction Relation

Step 3 **Definition:** Let  $v: VAR \longrightarrow \{T, F\}$ We say that v satisfies a formula  $A \in \mathcal{F}$  iff  $v^*(A) = T$ Notation:  $v \models A$ **Definition:** We say that v does not satisfy a formula  $A \in \mathcal{F}$  iff  $v^*(A) \neq T$ Notation:  $v \not\models A$ 

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The relation  $\models$  is called a **satisfaction relation** 

### Semantics: Satisfaction Relation

**Observe** that  $v^*(A) \neq T$  is is equivalent to the fact that  $v^*(A) = F$  only in 2-valued semantics and

 $v \not\models A$  iff  $v^*(A) = F$ 

#### Definition

We say that **v** falsifies the formula A iff  $v^*(A) = F$ Remark

For any formula  $A \in \mathcal{F}$ 

 $\mathbf{v} \not\models \mathbf{A}$  iff **v** falsifies the formula  $\mathbf{A}$ 

## Examples

**Example 1**: Let  $A = ((a \Rightarrow b) \cup \neg a))$  and  $v : VAR \longrightarrow \{T, F\}$  be such that v(a) = T, v(b) = F

We calculate  $v^*(A)$  using a **short hand notation** as follows

 $(T \Rightarrow F) \cup \neg T = F \cup F = F$ 

By definitiom

$$\mathsf{v} \not\models ((a \Rightarrow b) \cup \neg a))$$

Observe that we did not need to specify the v(x) of any  $x \in VAR - \{a, b\}$ , as these values do not influence the computation of the logical value  $v^*(A)$ 

### Examples

**Example 2** Let  $A = ((a \cap \neg b) \cup \neg c)$  and  $v : VAR \longrightarrow \{T, F\}$  be such that v(a) = T, v(b) = F, v(c) = TWe calculate  $v^*(A)$  using a **short hand notation** as follows

$$(T \cap \neg F) \cup \neg T = (T \cap T) \cup F = T \cup F = T$$

By definition

 $v \models ((a \cap \neg b) \cup \neg c)$ 

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### Examples

**Example 3** Let  $A = ((a \cap \neg b) \cup \neg c)$ Consider now  $v_1 : VAR \longrightarrow \{T, F\}$  such that  $v_1(a) = T, v_1(b) = F, v_1(c) = T$  and  $v_1(x) = F$ , for all  $x \in VAR - \{a, b, c\}$ Observe that  $v(a) = v_1(a), v(b) = v_1(b), v(c) = v_1(c)$ Hence we get

 $v_1 \models ((a \cap \neg b) \cup \neg c)$ 

### Examples

**Example 4** Let  $A = ((a \cap \neg b) \cup \neg c)$ Consider now  $v_2 : VAR \longrightarrow \{T, F\}$  such that  $v_2(a) = T, v_2(b) = F, v_2(c) = T, v_2(d) = T$  and  $v_1(x) = F$ , for all  $x \in VAR - \{a, b, c, d\}$ Observe that  $v(a) = v_2(a), v(b) = v_2(b), v(c) = v_2(c)$ Hence we get

 $v_2 \models ((a \cap \neg b) \cup \neg c)$ 

#### Semantics: Model, Counter-Model

#### Definition:

Given a formula  $A \in \mathcal{F}$  and  $v : VAR \longrightarrow \{T, F\}$ 

Any v such that  $v \models A$  is called a **model** for A

Any v such that  $v \not\models A$  is called a **counter model** for A

Observe that all truth assignments v,  $v_1$ ,  $v_2$  from our **Examples 2, 3, 4** are **models** for the same formula A

# Semantics: Tautology

# Step 4 Definition:

For any formula  $A \in \mathcal{F}$ 

A is a tautology iff  $v^*(A) = T$ , for all  $v : VAR \longrightarrow \{T, F\}$ 

# i.e. we have that

A is a tautology iff any  $v : VAR \longrightarrow \{T, F\}$  is a model for A

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#### Notation

We write symbolically  $\models A$  for the statement "A is a tautology"

Semantics: not a tautology

### Definition

A is not a tautology iff there is v, such that  $v^*(A) \neq T$ 

#### i.e. we have that

A is not a tautology iff A has a counter-model

#### Notation

We write  $\not\models A$  to denote the statement "A is not a tautology"

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# How Many

We just saw from the **Examples 2, 3, 4** that given a model v for a formula A, we defined 2 other models for A

These models were identical with v on the variables in the formula A

Visibly we can keep constructing in a similar way more and more of such models

A natural question arises:

Given a **model** for a the formula A, **how many other models** for A can be constructed?

The same question can be asked about counter-models for A, if they exist

**Challenge Problem** 

Challenge Problem : prove the following

#### **Model Theorem**

For any formula  $A \in \mathcal{F}$ ,

If A has a **model** (counter- model), then it has uncountably many (exactly as many as real numbers) of **models** (counter-models)

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# How Many

Here is a more general question

# Question

Given a formula  $A \in \mathcal{F}$ ,

how many truth assignments we have to consider to prove that the formula A? is a **tautology**?

We prove that there are as many of such truth assignments as real numbers

But FORTUNATELY only a finite number of them is differs on the variables included in the formula A and we do have the following

# Tautology DecidabilityTheorem

The notion of classical propositional tautology  $\models A$  is decidable

### **Restricted Truth Assignments**

To address and to answer these questions formally we first introduce some notations and definitions

**Notation:** for any formula *A*, we denote by  $VAR_A$  a set of all variables that appear in A

**Definition:** Given  $v : VAR \longrightarrow \{T, F\}$ , any function  $v_A : VAR_A \longrightarrow \{T, F\}$  such that  $v(a) = v_A(a)$  for all  $a \in VAR_A$  is called a **restriction** of v to the formula A **Fact 1** 

For any formula A, any v, and its restriction  $v_A$ 

$$v \models A$$
 iff  $v_A \models A$ 

**Restricted Model** 

**Definition:** Given a formula  $A \in \mathcal{F}$ , any function  $w : VAR_A \longrightarrow \{T, F\}$ 

is called a truth assignment restricted to A

**Definition** Given a formula  $A \in \mathcal{F}$ Any function

 $w: VAR_A \longrightarrow \{T, F\}$  such that  $w^*(A) = T$ is called a **restricted MODEL** for A

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### Example

Example  $A = ((a \cap \neg b) \cup \neg c)$   $VAR_A = \{a, b, c\}$ 

Truth assignment **restricted** to A is any function:

 $w: \{a, b, c\} \longrightarrow \{T, F\}.$ 

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We use the following theorem to count all possible truth assignment **restricted** to A

### **Counting Functions**

### **Counting Functions Theorem**

For any finite sets *A* and *B*, if the set *A* has n elements and *B* has m elements, then there are  $m^n$  possible functions that map *A* into *B* **Proof** by Mathematical Induction over m

#### Example:

There are  $2^3 = 8$  truth assignments **w** restricted to

$$A = ((a \Rightarrow \neg b) \cup \neg c)$$

**Counting Theorem** 

**Counting Theorem** 

For any  $A \in \mathcal{F}$ , there are

 $2^{|VAR_A|}$ 

possible truth assignments restricted to A

#### Example

Let  $A = ((a \cap \neg b) \cup \neg c)$ 

All w restricted to A are listed in the table below

w	а	b	С	w*(A) computation	w*(A)
w1	Т	Т	Т	$(T \Rightarrow T) \cup \neg T = T \cup F = T$	Т
W2	Т	Т	F	$(T \Rightarrow T) \cup \neg F = T \cup T = T$	т
w <sub>3</sub>	Т	F	F	$(T \Rightarrow F) \cup \neg F = F \cup T = T$	т
w4	F	F	Т	$(F \Rightarrow F) \cup \neg T = T \cup F = T$	Т
w <sub>5</sub>	F	Т	Т	$(F \Rightarrow T) \cup \neg T = T \cup F = T$	т
W <sub>6</sub>	F	Т	F	$(F \Rightarrow T) \cup \neg F = T \cup T = T$	т
W <sub>7</sub>	Т	F	Т	$(T \Rightarrow F) \cup \neg T = F \cup F = F$	F
w <sub>8</sub>	F	F	F	$(F \Rightarrow F) \cup \neg F = T \cup T = T$	Т

 $w_1, w_2, w_3, w_4w_5, w_6, w_8$  are restricted models for A  $w_7$  is a restricted counter- model for A

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# **Restrictions and Extensions**

Given a formula A and  $w : VAR_A \longrightarrow \{T, F\}$ Definition

Any function v, such that  $v : VAR \longrightarrow \{T, F\}$  and v(a) = w(a), for all  $a \in VAR_A$  is called an **extension** of w to the set *VAR* of all propositional variables

# Fact 2

For any formula A, any w restricted to A , and any of its extensions v

 $w \models A$  iff  $v \models A$ 

Tautology and Decidability

By the definition of a tautology and **Facts 1, 2** we get the following

TautologyTheorem

 $\models$  A iff  $w \models A$  for all  $w : VAR_A \longrightarrow \{T, F\}$ 

From above and the Counting Theorem we get

#### Tautology DecidabilityTheorem

The notion of classical propositional tautology  $\models A$  is decidable

### Tautology Verification

We just PROVED correctness of the well known

### Truth Table Tautology Verification Method :

to verify whether  $\models A$  list and evaluate all possible truth assignments w restricted to A and we have that

- $\models$  A if all w evaluate to T
- $\not\models$  A if there is one w that evaluates to F

### **Truth Table Example**

Consider a formula A:

 $(a \Rightarrow (a \cup b))$ 

We write the Truth Table:

w	а	b	w <sup>*</sup> (A) computation	w*(A)
w <sub>1</sub>	Т	Т	$(T \Rightarrow (T \cup T)) = (T \Rightarrow T) = T$	Т
w2	Т	F	$(T \Rightarrow (T \cup F)) = (T \Rightarrow T) = T$	Т
W <sub>3</sub>	F	Т	$(F \Rightarrow (F \cup T)) = (F \Rightarrow T) = T$	Т
w4	F	F	$(F \Rightarrow (F \cup F)) = (F \Rightarrow F) = T$	Т

We evaluated that for all **w** restricted to A, i.e. all functions  $w : VAR_A \longrightarrow \{T, F\}, w \models A$ 

This proves by TautologyTheorem

$$\models (a \Rightarrow (a \cup b))$$

### **Tautology Verification**

Imagine now that A has for example 200 variables.

To find whether A is a tautology by using the **Truth Table Method** one would have to evaluate 200 variables long expressions - not to mention that one would have to list 2<sup>200</sup> **restricted** truth assignments

I want you to use now and later in case of many valued semantics a more intelligent (and much faster!) method called **Proof by Contradiction Method** 

In fact, I will not accept the Truth Tables verifications on any TEST and students using it will get 0 pts for the problem Tautology - Proof by Contradiction Method

# **Proof by Contradiction Method:**

In this method, in order to **prove** that  $\models A$  we proceed as follows

We **assume** that  $\not\models A$ 

We work with this assumption

If we get a contradiction, we have proved that  $\not\models A$  is impossible

We hence **proved**  $\models$  **A** 

If we do not get a **contradiction**, it means that the assumption  $\not\models A$  is **true**, i.e.

we have **proved** that  $\not\models A$ 

Tautology - Proof by Contradiction Method

### **Proof by Contradiction Method:**

in order to verify whether  $\models A$  one works backwards, trying to find a truth assignment v which makes a formula A false. If we **find one**, it means that A is not a tautology if we prove that it is **impossible**, i.e. we got a **contradiction** it means that the formula is a tautology

# Example

Let  $A = (a \Rightarrow (a \cup b))$ 

**Step 1**: Assume that  $\not\models A$ , i.e. we write in a shorthand notion A = F

**Step 2**: We use shorthand notation to analyze **Strep 1**  $(a \Rightarrow (a \cup b)) = F$  iff a = T and  $(a \cup b) = F$ 

Step 3: Analyze Step 2

a = T and  $(a \cup b) = F$ , i.e.  $(T \cup b) = F$ 

This is **impossible** by the definition of  $\cup$ 

We got a contradiction, hence

 $\models (a \Rightarrow (a \cup b))$ 

# Example

**Observe** that exactly the same reasoning proves that for any formulas  $A, B \in \mathcal{F}$ ,

 $\models (A \Rightarrow (A \cup B))$ 

The following formulas are also tautologies

 $((((a \Rightarrow b) \cap \neg c) \Rightarrow ((((a \Rightarrow b) \cap \neg c) \cup \neg d)), \text{and} \\ ((((((a \Rightarrow b) \cap \neg c) \cup d) \cap \neg e) \Rightarrow ((((a \Rightarrow b) \cap \neg c) \cup d) \cap \neg e) \\ \cup (a \Rightarrow \neg e)))$ 

because they are particular cases of  $(A \Rightarrow (A \cup B))$ 

Tautologies, Contradictions

Set of all Tautologies

$$\mathbf{T} = \{ A \in \mathcal{F} : \models A \}$$

# Definition

A formula  $A \in \mathcal{F}$  is called a **contradiction** if it does not have a model

Contradiction Notation: = | A

Directly from the definition we have that

= |A| if and only f  $v \not\models A$  for all  $v : VAR \longrightarrow \{T, F\}$ 

Set of all Contradictions

$$\mathbf{C} = \{ A \in \mathcal{F} : = | A \}$$

#### Examples

Tautology $(A \Rightarrow (B \Rightarrow A))$ Contradiction $(A \cap \neg A)$ Neither $(a \cup \neg b)$ 

Consider the formula  $(a \cup \neg b)$ Any v such that v(a) = T is a **model** for  $(a \cup \neg b)$ , so it is not a contradiction Any v such that v(a) = F, v(b) = T is a **counter-model** for  $(a \cup \neg b)$  so  $\nvDash (a \cup \neg b)$ 

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### **Simple Properties**

**Theorem 1** For any formula  $A \in \mathcal{F}$  the following conditions are equivalent.

- (1) **A** ∈ **T**
- (2) **¬***A* ∈ **C**
- (3) For all  $\mathbf{v}, \mathbf{v} \models \mathbf{A}$

**Theorem 2** For any formula  $A \in \mathcal{F}$  the following conditions are equivalent.

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- (1) **A** ∈ **C**
- (2)  $\neg A \in T$
- (6) For all v,  $v \not\models A$

### **Constructing New Tautologies**

We now formulate and prove a theorem which describes validity of a method of constructing **new tautologies** from **given tautologies** 

First we introduce some convenient notations.

**Notation 1**: for any  $A \in \mathcal{F}$  we write

 $A(a_1, a_2, ..., a_n)$ 

to denote that  $a_1, a_2, \dots a_n$  are fall propositional variables appearing in *A* 

**Notation 2**: let  $A_1, ..., A_n$  be any formulas, we write

 $A(a_1/A_1, ..., a_n/A_n)$ 

to denote the result of **simultaneous replacement** (substitution) all variables  $a_1, a_2, ..., a_n$  in A by formulas  $A_1, ..., A_n$ , respectively.

#### Constructing NewTautologies

**Theorem** For any formulas  $A, A_1, ..., A_n \in \mathcal{F}$ , IF  $\models A(a_1, a_2, ..., a_n)$  and  $B = A(a_1/A_1, ..., a_n/A_n)$ , THEN  $\models B$ 

**Proof**: Let  $B = A(a_1/A_1, ..., a_n/A_n)$  and let  $b_1, b_2, ..., b_m$  be all propositional variables which occur in  $A_1, ..., A_n$ Given a truth assignment  $v : VAR \longrightarrow \{T, F\}$ , the values  $v(b_1), v(b_2), ..., v(b_m)$  define  $v^*(A_1), ..., v^*(A_n)$  and, in turn define  $v^*(A(a_1/A_1, ..., a_n/A_n))$ 

#### Constructing NewTautologies

Let now  $w : VAR \longrightarrow \{T, F\}$  be a truth assignment such that  $w(a_1) = v^*(A_1), w(a_2) = v^*(A_2), ...w(a_n) = v^*(A_n).$ Obviously,  $v^*(B) = w^*(A).$ Since  $\models A, w^*(A) = T$ , for all possible w, hence  $v^*(B) = w^*(A) = T$  for all truth assignments w and we have  $\models B$ 

### Models for Sets of Formulas

Consider  $\mathcal{L} = \mathcal{L}_{CON}$  and let  $\mathcal{S} \neq \emptyset$  be any non empty set of formulas of  $\mathcal{L}$ , i.e.

 $\mathcal{S} \subseteq \mathcal{F}$ 

We adopt the following definition.

# Definition

A truth truth assignment  $v : VAR \longrightarrow \{T, F\}$ 

is a model for the set S of formulas if and only if

 $v \models A$  for all formulas  $A \in S$ 

We write

# $v \models S$

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to denote that v is a model for the set S of formulas

Counter- Models for Sets of Formulas

Similarly, we define a notion of a **counter-model Definition** 

A truth assignment  $v : VAR \longrightarrow \{T, F\}$ 

is a counter-model for the set  $S \neq \emptyset$ 

of formulas if and only if

# $v \not\models A$ for some formula $A \in S$

We write

### $v \not\models S$

to denote that v is a **counter-model** for the set S of formulas

Restricted Model for Sets of Formulas

### **Remark** that the set S can be infinite, or finite

In a case when S is a **finite** subset of formulas we define, as before, a notion of restricted model and restricted counter-model.

#### Definition

Let *S* be a **finite** subset of formulas and  $v \models S$ Any restriction of the model v to the domain

$$VAR_{\mathcal{S}} = \bigcup_{A \in \mathcal{S}} VAR_{A}$$

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is called a restricted model for  $\mathcal{S}$ 

Restricted Counter - Model for Sets of Formulas

### Definition

Any restriction of a **counter-model** v of a set  $S \neq \emptyset$  of formulas to the domain

$$VAR_{S} = \bigcup_{A \in S} VAR_{A}$$

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is called a restricted counter-model for  $\mathcal{S}$ 

#### Example

Example Let  $\mathcal{L} = \mathcal{L}_{\{\neg, \cap\}}$  and let  $S = \{a, (a \cap \neg b), c, \neg b\}$ We have now  $VAR_{S} = \{a, b, c\}$ and  $v: VAR_S \rightarrow \{T, F\}$  such that v(a) = T, v(c) = T, v(b) = F is a restricted model for S and  $v: VAR_S \rightarrow \{T, F\}$  such that v(a) = Fis a restricted counter-model for S

# Models for Infinite Sets

The set S from the previous example was a finite set Natural question arises:

# Question

Give an example of an infinite set *S* that has a model

Give an example of an infinite set  $\mathcal{S}$  that does not have model

Ex1 Consider set T of all tautologies

It is a countably infinite set and by definition of a tautology any v is a **model** for **T**, i.e.  $v \models T$ 

Ex2 Consider set C of all contradictions

It is a countably infinite set and

for any v,  $v \not\models C$  by definition of a contradiction, i.e. any any v is a **counter-model** for C

### **Challenge Problems**

- **P1** Give an example of an infinite set S, such that  $S \neq T$  and S has a model
- **P2** Give an example of an infinite set S, such that  $S \cap T = \emptyset$  and S has a model
- **P3** Give an example of an infinite set S, such that  $S \neq C$  and S does not have a model
- **P4** Give an example of an infinite set S, such that  $S \neq C$  and S has a counter model
- **P5** Give an example of an infinite set S, such that  $S \cap C = \emptyset$  and S has a counter model
### Chapter 4: Consistent Sets of Formulas

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### Definition

A set  $\mathcal{G} \subseteq \mathcal{F}$  of **formulas** is called **consistent** if and only if  $\mathcal{G}$  has a model, i.e. we have that

 $\mathcal{G} \subseteq \mathcal{F}$  is **consistent** if and only if **there is v** such that  $\mathbf{v} \models \mathcal{G}$ 

Otherwise *G* is called **inconsistent** 

# HALF Challenge Problems

- **P6** Give an example of an infinite set S, such that  $S \neq T$  and S is **consistent**
- **P7** Give an example of an infinite set *S*, such that
- $S \cap T = \emptyset$  and S is consistent
- **P8** Give an example of an infinite set S, such that  $S \neq C$  and S is **inconsistent**

**P9** Give an example of an infinite set S, such that  $S \cap C = \emptyset$  and S is **inconsistent** 

### Chapter 4: Independent Statements

### Definition

A formula A is called **independent** from a set  $\mathcal{G} \subseteq \mathcal{F}$ if and only if **there are** truth assignments  $v_1, v_2$  such that

 $v_1 \models \mathcal{G} \cup \{A\}$  and  $v_2 \models \mathcal{G} \cup \{\neg A\}$ 

i.e. we say that a formula A is **independent** if and only if

 $\mathcal{G} \cup \{A\}$  and  $\mathcal{G} \cup \{\neg A\}$  are consistent

# Example

### Example

Given a set

$$\mathcal{G} = \{((a \cap b) \Rightarrow b), (a \cup b), \neg a\}$$

Show that  $\mathcal{G}$  is **consistent** Solution We have to find  $v : VAR \longrightarrow \{T, F\}$  such that

 $v \models G$ 

It means that we need to find v such that

 $v^*((a \cap b) \Rightarrow b) = T$ ,  $v^*(a \cup b) = T$ ,  $v^*(\neg a) = T$ 

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### Consistent: Example

**1.** Formula  $((a \cap b) \Rightarrow b)$  is a tautology, i.e.  $v^*((a \cap b) \Rightarrow b) = T$  for any v and we do not need to consider it anymore.

**2.** Formula  $\neg a = T$  (we use shorthand notation) if and only if a = F so we get that v must be such that v(a) = F

**2.** We want  $(a \cup b) = T$  but v is such that v(a) = F so  $(a \cup b) = F \cup b = T$  if and only if b = T

This **means** that for any  $v : VAR \longrightarrow \{T, F\}$  such that v(a) = F, v(b) = T

$$v \models G$$

and we proved that  $\mathcal{G}$  is consistent

### Example

Show that a formula  $A = ((a \Rightarrow b) \cap c)$  is independent of

$$\mathcal{G} = \{((a \cap b) \Rightarrow b), (a \cup b), \neg a\}$$

### Solution

We construct  $v_1$ ,  $v_2$ :  $VAR \longrightarrow \{T, F\}$  such that

$$v_1 \models \mathcal{G} \cup \{A\}$$
 and  $v_2 \models \mathcal{G} \cup \{\neg A\}$ 

We have just proved that any  $v : VAR \longrightarrow \{T, F\}$  such that v(a) = F, v(b) = T is a **model** for  $\mathcal{G}$ 

#### Independent: Example

Take as  $v_1$  any truth assignment such that  $v_1(a) = v(a) = F$ ,  $v_1(b) = v(b) = T$ ,  $v_1(c) = T$ We evaluate  $v_1^*(A) = v_1^*((a \Rightarrow b) \cap c) = (F \Rightarrow T) \cap T = T$ This proves that  $v_1 \models \mathcal{G} \cup \{A\}$ 

Take as  $v_2$  any truth assignment such that  $v_2(a) = v(a) = F$ ,  $v_2(b) = v(b) = T$ ,  $v_2(c) = F$ We evaluate  $v_2^*(\neg A) = v_2^*(\neg(a \Rightarrow b) \cap c)) = T \cap T = T$ This proves that  $v_2 \models \mathcal{G} \cup \{\neg A\}$ 

It ends the proof that A is **independent** of  $\mathcal{G}$ 

## Not Independent: Example

### Example

Show that a formula  $A = (\neg a \cap b)$  is **not independent** of

$$\mathcal{G} = \{((a \cap b) \Rightarrow b), (a \cup b), \neg a\}$$

### Solution

We have to show that it is impossible to construct  $v_1, v_2$  such that

 $v_1 \models \mathcal{G} \cup \{A\}$  and  $v_2 \models \mathcal{G} \cup \{\neg A\}$ 

**Observe** that we have just proved that any v such that v(a) = F, and v(b) = T is the only model restricted to the set of variables  $\{a, b\}$  for  $\mathcal{G}$  and  $\{a, b\} = VAR_A$ So we have to check now if it is **possible**  $v \models A$  and  $v \models \neg A$ 

### Not Independent: Example

We have to evaluate  $v^*(A)$  and  $v^*(\neg A)$  for v(a) = F, and v(b) = T  $v^*(A) = v^*((\neg a \cap b) = \neg v(a) \cap v(b) = \neg F \cap T = T \cap T = T$ and so  $v \models A$   $v^*(\neg A) = \neg v^*(A) = \neg T = F$ and so  $v \not\models \neg A$ 

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This end the proof that A is **not independent** of  $\mathcal{G}$ 

### Independent: Another Example

## Example

Given a set  $\mathcal{G} = \{a, (a \Rightarrow b)\}$ , find a formula A that is **independent** from  $\mathcal{G}$ 

**Observe** that v such that v(a) = T, v(b) = T is the only restricted model for G

So we have to come up with a formula A such that there are two different truth assignments,  $v_1$  and  $v_2$ , and

 $v_1 \models \mathcal{G} \cup \{A\}$  and  $v_2 \models \mathcal{G} \cup \{\neg A\}$ 

Let's consider A = c, then  $\mathcal{G} \cup \{A\} = \{a, (a \Rightarrow b), c\}$ A truth assignment  $v_1$ , such that  $v_1(a) = T$ ,  $v_1(b) = T$  and  $v_1(c) = T$  is a **model** for  $\mathcal{G} \cup \{A\}$ Likewise for  $\mathcal{G} \cup \{\neg A\} = \{a, (a \Rightarrow b), \neg c\}$ Any  $v_2$ , such that  $v_2(a) = T$ ,  $v_2(b) = T$  and  $v_2(c) = F$  is a **model** for  $\mathcal{G} \cup \{\neg A\}$  and so the formula A is **independent**  Challenge Problem

**Challenge Problem** 

Find an infinite number of formulas that are independent of a set

 $\mathcal{G} = \{((a \cap b) \Rightarrow b), (a \cup b), \neg a\}$ 

# Challenge Problem Solution

This my solution - there are many others- this one seemed to me the **most simple** 

# Solution

We just proved that any v such that v(a) = F, v(b) = T is the only model restricted to the set of variables  $\{a, b\}$  and so all other possible models for *G* must be **extensions** of v

### **Challenge Problem Solution**

We **define** a countably infinite set of formulas (and their negations) and corresponding **extensions** of v (restricted to to the set of variables  $\{a, b\}$ ) such that  $v \models G$  as follows **Observe** that **all extensions** of v restricted to to the set of variables  $\{a, b\}$  have as domain the infinitely countable set

 $VAR = \{a_1, a_2, ..., a_n...\}$ 

We take as an infinite set of formulas in which every formula independent of  $\mathcal{G}$  the set of atomic formulas

$$\mathcal{F}_0 = \{a_1, a_2, \ldots, a_n, \ldots\} - \{a, b\}$$

### **Challenge Problem Solution**

Let  $c \in \mathcal{F}_0 = \{a_1, a_2, \dots, a_n \dots\} - \{a, b\}$ 

We define truth assignments  $v_1, v_2 : VAR \longrightarrow \{T, F\}$  such that

 $v_1 \models \mathcal{G} \cup \{c\}$  and  $v_2 \models \mathcal{G} \cup \{\neg c\}$ 

as follows

 $v_1(a) = v(a) = F$ ,  $v_1(b) = v(b) = T$  and  $v_1(c) = T$  for any  $c \in \mathcal{F}_0$  $v_2(a) = v(a) = F$ ,  $v_2(b) = v(b) = T$  and  $v_2(c) = F$  for any  $c \in \mathcal{F}_0$ 

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