CSE581
Computer Science Fundamentals: Theory

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P1 LOGIC: LECTURE 3

Chapter 3
Propositional Languages

PART 1: Propositional Languages: Intuitive Introduction

PART 2: Propositional Languages: Formal Definitions

PART 1: Propositional Languages Intuitive Introduction

We define now a general notion of a propositional language.

We show how to obtain, as specific cases, various languages for propositional classical logic and some non-classical logics

We assume the following:

All propositional languages contain an infinitely countable set of variables VAR, which elements are denoted by

$$a, b, c, ...$$

with indices, if necessary

All propositional languages share the general way their sets of formulas are formed

Propositional Languages

We distinguish one propositional language from the other is the choice of its set of propositional connectives.

We adopt a notation

$$\mathcal{L}_{CON}$$

where CON stands for the set of connectives

We use a notation

$$\mathcal{L}$$

when the set of connectives is fixed

Propositional Languages

For example, the language

$$\mathcal{L}_{\{\neg\}}$$

denotes a propositional language with only one connective $\neg$

The language

$$\mathcal{L}_{\{\neg,\Rightarrow\}}$$

denotes that a language with two connectives $\neg$ and $\Rightarrow$ adopted as propositional connectives

Remember: formal languages deal with symbols only and are also called symbolic languages

General Principles

Symbols for connectives do have intuitive meaning.

Semantics provides a formal meaning of the connectives and is defined separately.

One language can have many semantics.

Different logics can share the same language.

For example: the language

$$\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$$

is used as a propositional language of classical and intuitionistic logics, some many- valued logics, and we extend it to the language of many modal logics

General Principles

Several languages can share the same semantics.

The classical propositional logic is the best example of such situation.

Due to the **functional dependency** of **classical logic connectives** the languages:

$$\mathcal{L}_{\{\neg, \Rightarrow\}}, \quad \mathcal{L}_{\{\neg, \cap\}}, \quad \mathcal{L}_{\{\neg, \cup\}}, \quad \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}},$$

$$\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow, \Leftrightarrow\}}, \quad \mathcal{L}_{\{\uparrow\}}, \quad \mathcal{L}_{\{\downarrow\}}$$

are all **equivalent** under the classical semantics

We will **define formally** the equivalency of languages in the next lecture

General Principles

Propositional connectives have well established **names** and the way we read them, even if their semantics may differ

We use names **negation, conjunction, disjunction** and **implication** for $\neg$, $\cap$, $\cup$, $\Rightarrow$, respectively

The connective $\uparrow$ is called **alternative negation** and $A \uparrow B$ reads: not both A and B

The connective $\downarrow$ is called **joint negation** and $A \downarrow B$ reads: neither A nor B

Some Non-Classical Propositional Connectives

Other most common propositional connectives are **modal** connectives of **possibility** and **necessity**

Modal connectives are not extensional

Standard **modal symbols** are:
$\Box$ for **necessity** and $\Diamond$ for **possibility**.

We will also use symbols C and I for modal connectives of possibility and necessity, respectively.

Some Non-Classical Propositional Connectives

The formula CA, or $\Diamond A$ reads:

it is **possible** that A or A is **possible**

The formula IA, or $\Box A$ reads:

it is **necessary** that A or A is **necessary**

Modal Propositional Connectives

Symbols C and I are used for their topological meaning in the semantics for the standard **modal logics** S4 and S5

In topology C is a symbol for a set closure operation

CA means a **closure** of a set A

I is a symbol for a set **interior** operation

IA denotes an interior of the set A

Modal logics extend the **classical logic**

A modal logic languages are for example

$$\mathcal{L}_{\{C, I, \neg, \cap, \cup, \Rightarrow\}} \quad \text{or} \quad \mathcal{L}_{\{\Box, \Diamond, \neg, \cap, \cup, \Rightarrow\}}$$

Some More Non-Extensional Connectives

Knowledge logics also extend the classical logic by adding a new one argument knowledge connective

The knowledge connective is often denoted by K

A formula KA reads: it is **known** that A or A is **known**

A language of a **knowledge logic** is for example

$$\mathcal{L}_{\{K, \neg, \cap, \cup, \Rightarrow\}}$$

Some More Non-Extensional Connectives

Autoepistemic logics extend classical logic by adding an one argument believe connective, often denoted by B

A formula BA reads: it is **believed** that A

A language of an autoepistemic logic is for example

$$\mathcal{L}_{\{B, \neg, \cap, \cup, \Rightarrow\}}$$

Some More Non-Extensional Connectives

Temporal logics also extend classical logic by adding one argument temporal connectives

Some of temporal connectives are: F, P, G, H

Their intuitive meanings are:

FA reads A is true at **some future** time,

PA reads A was true at **some past** time,

GA reads A will be true at **all future** times

HA reads A has **always** been true in the **past**

Propositional Connectives

It is possible to create connectives with more then **one** or **two** arguments

We consider here only one or two argument connectives

Chapter 3
Propositional Languages
PART 2: Formal Definitions

Propositional Language

Definition

A **propositional language** is a pair

$$\mathcal{L} = (\mathcal{A}, \mathcal{F})$$

where $\mathcal{A}, \mathcal{F}$ are called respectively an **alphabet** and a **set of formulas**,

Definition

Alphabet is a set

$$\mathcal{A} = VAR \cup CON \cup PAR$$

VAR, CON, PAR are all disjoint sets of propositional **variables, connectives** and **parenthesis**, respectively

The sets VAR, CON are non-empty

Alphabet Components

VAR is a countably infinite set of **propositional variables**

We denote elements of VAR by

$$a, b, c, d, \ldots$$

with indices if necessary

$CON \neq \emptyset$ is a finite set of **propositional connectives**

We assume that the set CON of logical connectives is non-empty, i.e. that a propositional language always has at least one connective.

Alphabet Components

Notation

We **denote** the language $\mathcal{L}$ with the set of connectives CON by

$$\mathcal{L}_{CON}$$

Observe that propositional languages **differ** only on a **choice** of the propositional connectives, hence our notation.

Alphabet Components

PAR is a set of **auxiliary symbols**

This set may be empty; for example in case of Polish notation.

Assumptions

We assume here that PAR contains only 2 **parenthesis** and

$$PAR = \{(,)\}$$

We also assume that the set CON of logical connectives contains **only** **unary** and **binary** connectives, i.e.

$$CON = C_1 \cup C_2$$

where C_1 is the set of all unary connectives, and C_2 is the set of all binary connectives

Definition

The set $\mathcal{F}$ of **all formulas** of a propositional language $\mathcal{L}_{CON}$ is build **recursively** from the elements of the alphabet $\mathcal{A}$ as follows.

$\mathcal{F} \subseteq \mathcal{A}^*$ and $\mathcal{F}$ is the **smallest** set for which the following conditions are satisfied

(1) $VAR \subseteq \mathcal{F}$
(2) If $A \in \mathcal{F}$, $\triangledown \in C_1$, then $\triangledown A \in \mathcal{F}$
(3) If $A, B \in \mathcal{F}$, $\circ \in C_2$ i.e $\circ$ is a two argument connective, then
$(A \circ B) \in \mathcal{F}$

By (1) propositional variables are formulas and they are called **atomic formulas**

The set $\mathcal{F}$ is also called a set of all **well formed formulas** (wff) of the language $\mathcal{L}_{CON}$

Set of Formulas

Observe that the the alphabet $\mathcal{A}$ is countably infinite

Hence the set $\mathcal{A}^*$ of all finite sequences of elements of $\mathcal{A}$ is also countably infinite

By definition $\mathcal{F} \subseteq \mathcal{A}^*$ and hence we get that the set of all formulas $\mathcal{F}$ is also countably infinite

We state as separate fact

Fact

For any propositional language $\mathcal{L} = (\mathcal{A}, \mathcal{F})$, its sets of formulas $\mathcal{F}$ is always a **countably infinite** set

We hence **consider** here **only** infinitely countable languages

Main Connectives and Direct Sub-Formulas

$\triangledown$ is called a main connective of the formula $\triangledown A \in \mathcal{F}$

A is called its direct sub-formula of $\triangledown A$

$\circ$ is called a main connective of the formula $(A \circ B) \in \mathcal{F}$

A, B are called direct sub-formulas of $(A \circ B)$

Examples

E1 Main connective of $(a \Rightarrow \neg Nb)$ is $\Rightarrow$

$a, \neg Nb$ are direct sub-formulas

E2 Main connective of $N(a \Rightarrow \neg b)$ is N

$(a \Rightarrow \neg b)$ is the direct sub-formula

E3 Main connective of $\neg(a \Rightarrow \neg b)$ is $\neg$

$(a \Rightarrow \neg b)$ is the direct sub-formula

E4 Main connective of of $(\neg a \cup \neg(a \Rightarrow b))$ is $\cup$

$\neg a$, $\neg(a \Rightarrow b))$ are direct sub-formulas

Sub-Formulas

We define a notion of a sub-formula in two steps:

Step 1

For any formulas A and B, the formula A is a proper sub-formula of B if there is sequence of formulas, beginning with A, ending with B, and in which each term is a direct sub-formula of the next

Step 2

A sub-formula of a given formula A is any proper sub-formula of A, or A itself

Sub-Formulas Example

The formula $(\neg a \cup \neg(a \Rightarrow b))$
has two direct sub-formulas: $\neg a, \quad \neg(a \Rightarrow b)$

The direct sub-formulas of $\neg a, \quad \neg(a \Rightarrow b)$

are respectively $a, \quad (a \Rightarrow b)$

The direct sub-formulas of $a, (a \Rightarrow b)$, are a, b

END of the process

Example

Given a formula

$$(\neg a \cup \neg(a \Rightarrow b))$$

Its set of all proper sub-formulas is:

$$S = \{\neg a, \neg(a \Rightarrow b), a, (a \Rightarrow b), b\}$$

The set of **all** its sub-formulas is

$$S \cup \{(\neg a \cup \neg(a \Rightarrow b))\}$$

Formula Degree Definition

We **define** a degree of a formula as a number of occurrences of logical connectives in the formula.

Example

The degree of $(\neg a \cup \neg(a \Rightarrow b))$ is 4

The degree of $\neg(a \Rightarrow b))$ is 2

The degree of $\neg a$ is 1

The degree of a is 0

Formula Degree

A degree of a formula is number of occurrences of logical connectives in the formula

Observation: the degree of any proper sub-formula of A must be one less than the degree of A
This is the central fact upon which mathematical induction arguments are based

Proofs of properties of formulas are usually carried by mathematical induction on their degrees

Exercise

Exercise 1

Consider a language

$$\mathcal{L} = \mathcal{L}_{\{\neg,\ \Diamond,\ \Box,\ \cup,\ \cap,\ \Rightarrow\}}$$

and a set $S \subseteq \mathcal{A}^*$ such that

$$S = \{\Diamond\neg a \Rightarrow (a \cup b), (\Diamond(\neg a \Rightarrow (a \cup b))), \\ \Diamond\neg(a \Rightarrow (a \cup b))\}$$

1. Determine which of the elements of S are, and which are not well formed formulas (wff) of $\mathcal{L}$
2. If a formula A is a well formed formula, i.e. $A \in \mathcal{F}$, determine its its main connective.
3. If $A \notin \mathcal{F}$ write the correct formula and then determine its **main connective**

Exercise 1 Solution

Solution

The formula $\Diamond\neg a \Rightarrow (a \cup b)$ **is not a well formed formula**

The correct formula is

$$(\Diamond\neg a \Rightarrow (a \cup b))$$

The main connective is $\Rightarrow$

The correct formula says:

If negation of a is possible, then we have a or b

Another correct formula in is

$$\Diamond(\neg a \Rightarrow (a \cup b))$$

The main connective is $\Diamond$

The corrected formula says:

It is possible that not a implies a or b

Exercise 1 Solution

The formula $(\Diamond(\neg a \Rightarrow (a \cup b)))$ **is not correct**

The correct formula is

$$\Diamond(\neg a \Rightarrow (a \cup b))$$

The main connective is $\Diamond$

The correct formula says:

It is possible that not a implies a or b

$\Diamond\neg(a \Rightarrow (a \cup b))$ is a **correct formula**

The main connective is $\Diamond$

The formula says:

It is possible that it is not true that a implies a or b

Exercise 2

Given a formula:

$$\Diamond((a \cup \neg a) \cap b)$$

1. Determine its degree
2. Write down all its sub-formulas

Solution:

The degree is 4

All sub-formulas are:

$$\Diamond((a \cup \neg a) \cap b),\ ((a \cup \neg a) \cap b),$$

$$(a \cup \neg a),\ \neg a,\ b,\ a$$

Language Defined by a set S

Definition

Given a set S of formulas of a language $\mathcal{L}_{CON}$

Let $CS \subseteq CON$ be the set of **all connectives** that appear in formulas of S

A language $\mathcal{L}_{CS}$
is called the **language defined** by the set of formulas S

Example

Let S be a set
$S = \{((a \Rightarrow \neg b) \Rightarrow \neg a), \ \Box(\neg\Diamond a \Rightarrow \neg a)\}$

All connectives appearing in the formulas in S are:

$$\Rightarrow, \quad \neg, \quad \Box, \quad \Diamond$$

The **language defined** by the set S is

$$\mathcal{L}_{\{\neg, \ \Rightarrow, \ \Box, \ \Diamond\}}$$

Exercise 3

Write the following natural language statement:

From the fact that it is possible that Anne is not a boy we deduce that it is not possible that Anne is not a boy or, if it is possible that Anne is not a boy, then it is not necessary that Anne is pretty

in the following two ways

1. As a formula

$A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \Box, \Diamond, \cap, \cup, \Rightarrow\}}$

2. As a formula

$A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$

Exercise 3 Solution

1.We translate our statement into a formula

$A_1 \in \mathcal{F}_1$ of the language $\mathcal{L}_{\{\neg, \Box, \Diamond, \cap, \cup, \Rightarrow\}}$ as follows

Propositional Variables: a,b

a denotes statement: *Anne is a boy*,

b denotes a statement: *Anne is pretty*

Propositional Modal Connectives: $\Box, \Diamond$

$\Diamond$ denotes statement: *it is possible that*

$\Box$ denotes statement: *it is necessary that*

Translation 1: the formula A_1 is

$$(\Diamond\neg a \Rightarrow (\neg\Diamond\neg a \cup (\Diamond\neg a \Rightarrow \neg\Box b)))$$

Exercise 3 Solution

2. We translate our statement into a formula $A_2 \in \mathcal{F}_2$ of the language $\mathcal{L}_{\{\neg,\ \cap,\ \cup,\ \Rightarrow\}}$ as follows

Propositional Variables: a,b

a denotes statement: *it is possible that Anne is not a boy*

b denotes a statement: *it is necessary that Anne is pretty*

Translation 2: the formula A_2 is

$$(a \Rightarrow (\neg a \cup (a \Rightarrow \neg b)))$$

Exercise

Exercise 4

Write the following natural language statement:

For all natural numbers $n \in N$ the following implication holds: if $n < 0$, then there is a natural number m, such that it is possible that $n + m < 0$, OR it is not possible that there is a natural number m, such that $m > 0$

in the following two ways

1. As a formula

$A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$

2. As a formula

$A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \Box, \Diamond, \cap, \cup, \Rightarrow\}}$

Exercise 4 Solution

1. We translate our statement into a formula $A_1 \in \mathcal{F}_1$ of the language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$ as follows

Propositional Variables: a, b

a denotes statement: *For all natural numbers $n \in N$ the following implication holds: if $n < 0$, then there is a natural number m, such that it is possible that $n + m < 0$*

b denotes a statement: *it is possible that there is a natural number m, such that $m > 0$*

Translation: the formula A_1 is

$$(a \cup \neg b)$$

Exercise 4 Solution

2. We translate our statement into a formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg,\ \Box,\ \Diamond,\ \cap,\ \cup,\ \Rightarrow\}}$ as follows

Propositional Variables: a, b

a denotes statement: *For all natural numbers $n \in N$ the following implication holds: if $n < 0$, then there is a natural number m, such that it is possible that $n + m < 0$*

b denotes a statement: *there is a natural number m, such that $m > 0$*

Translation: the formula A_2 is

$$(a \cup \neg\Diamond b)$$

Exercise

Exercise 5

Write the following natural language statement:

The following statement holds for all natural numbers $n \in N$:

if $n < 0$, then there is a natural number m, such that it is possible that $n + m < 0$, OR it is not possible that there is a natural number m, such that $m > 0$

in the following two ways

1. As a formula

$A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$

2. As a formula

$A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \Box, \Diamond, \cap, \cup, \Rightarrow\}}$

Exercise

Exercise 6

Write the following natural language statement:

From the fact that each natural number is greater than zero we deduce that it is not possible that Anne is a boy or, if it is possible that Anne is not a boy, then it is necessary that it is not true that each natural number is greater than zero

in the following two ways

1. As a formula

$A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg,\ \Box,\ \Diamond,\ \cap,\ \cup,\ \Rightarrow\}}$

2. As a formula

$A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg,\ \cap,\ \cup,\ \Rightarrow\}}$