cse581
Computer Science Fundamentals: Theory

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TCB - LECTURE 1

TCB - THEORY OF COMPUTATION BASICS
PART 3: Special types of Binary Relations
PART 4: Finite and Infinite Sets
PART 5: Some Fundamental Proof Techniques

Theory of Computation BASICS
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Theory of Computation Basics

PART 6: Closures and Algorithms
Closures - Intuitive

Idea

Natural numbers $\mathbb{N}$ are closed under $+$, i.e. for given two natural numbers $n, m$ we always have that $n + m \in \mathbb{N}$

Natural numbers $\mathbb{N}$ are not closed under subtraction $-$, i.e. there are two natural numbers $n, m$ such that $n - m \notin \mathbb{N}$, for example $1, 2 \in \mathbb{N}$ and $1 - 2 \notin \mathbb{N}$

Integers $\mathbb{Z}$ are closed under $-$, moreover $\mathbb{Z}$ is the smallest set containing $\mathbb{N}$ and closed under subtraction $-$

The set $\mathbb{Z}$ is called a closure of $\mathbb{N}$ under subtraction $-$
Consider the two directed graphs $R$ (a) and $R^*$ (b) as shown below.

Observe that $R^* = R \cup \{(a_i, a_i) : i = 1, 2, 3, 4\} \cup \{(a_2, a_4)\}$, $R \subseteq R^*$ and is $R^*$ is reflexive and transitive whereas $R$ is neither, moreover $R^*$ is also the smallest set containing $R$ that is reflexive and transitive.

We call such relation $R^*$ the reflexive, transitive closure of $R$.

We define this concept formally in two ways and prove the equivalence of the two definitions.
Two Definitions of $R^*$

**Definition 1** of $R^*$

$R^*$ is called a reflexive, transitive closure of $R$ iff $R \subseteq R^*$ and is reflexive and transitive and is the smallest set with these properties.

This definition is based on a notion of a **closure property** which is any property of the form ”the set $B$ is closed under relations $R_1, R_2, \ldots, R_m$”

We define it formally and prove that **reflexivity** and **transitivity** are closures properties.

Hence we **justify** the name: reflexive, transitive closure of $R$ for $R^*$.
Two Definitions of $R^*$

Definition 2 of $R^*$
Let $R$ be a binary relation on a set $A$
The reflexive, transitive closure of $R$ is the relation

$$R^* = \{(a, b) \in A \times A : \text{there is a path from } a \text{ to } b \text{ in } R\}$$

This is a much simpler definition- and algorithmically more interesting as it uses a simple notion of a path
We hence start our investigations from it- and only later introduce all notions needed for the Definition 1 in order to prove that the $R^*$ defined above is really what its name says: the reflexive, transitive closure of $R$
Definition 2 of $R^*$

We bring back the following

**Path Definition**

A **path** in the binary relation $R$ is a finite sequence $a_1, \ldots, a_n$ such that $(a_i, a_{i+1}) \in R$, for $i = 1, 2, \ldots n - 1$ and $n \geq 1$

The path $a_1, \ldots, a_n$ is said to be from $a_1$ to $a_n$

The path $a_1$ (case when $n = 1$) always exist and is called a **trivial path** from $a_1$ to $a_1$

**Definition 2**

Let $R$ be a binary relation on a set $A$

The **reflexive, transitive closure of $R$** is the relation

$$R^* = \{(a, b) \in A \times A : \text{ there is a path from } a \text{ to } b \text{ in } R \}$$
Algorithms

Definition 2 immediately suggests an following algorithm for computing the reflexive transitive closure $R^*$ of any given binary relation $R$ over some finite set $A = \{a_1, a_2, \ldots, a_n\}$

Algorithm 1
Initially $R^* := 0$
for $i = 1, 2, \ldots, n$ do
for each $i$-tuple $(b_1, \ldots, b_i) \in A^i$ do
if $b_1, \ldots, b_i$ is a path in $R$ then add $(b_1, b_n)$ to $R^*$
Algorithms

We also have a following much faster algorithm

Algorithm 2
Initially \( R^* := R \cup \{(a_i, a_i) : a_i \in A\} \)
for \( j = 1, 2, \ldots, n \) do
for \( i = 1, 2, \ldots, n \) and \( k = 1, 2, \ldots, n \) do
if \( (a_i, a_j), (a_j, a_k) \in R^* \) but \( (a_i, a_k) \notin R^* \)
then add \( (a_i, a_k) \) to \( R^* \)
Closure Property Formal

We introduce now formally a concept of a closure property of a given set

**Definition**

Let $D$ be a set, let $n \geq 0$ and let $R \subseteq D^{n+1}$ be a $(n+1)$-ary relation on $D$

Then the subset $B$ of $D$ is said to be **closed under** $R$ if $b_{n+1} \in B$ whenever $(b_1, \ldots, b_n, b_{n+1}) \in R$

Any property of the form "the set $B$ is closed under relations $R_1, R_2, \ldots, R_m"$ is called a **closure property** of $B$
Closure Property Examples

Observe that any function \( f : D^n \rightarrow D \) is a special relation \( f \subseteq D^{n+1} \) so we have also defined what does it mean that a set \( A \subseteq D \) is **closed under** the function \( f \).

E1: \( + \) is a **closure property** of \( \mathbb{N} \).

Addition is a function \( + : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) defined by a formula \( + (n, m) = n + m \), i.e. it is a **relation** \( + \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N} \) such that

\[
+ = \{(n, m, n + m) : \ n, m \in \mathbb{N}\}
\]

Obviously the set \( \mathbb{N} \subseteq \mathbb{N} \) is (formally) **closed under** \( + \) because for any \( n, m \in \mathbb{N} \) we have that \( (n, m, n + m) \in + \).
Closures Property Examples

**E2:** \(\cap\) is a closure property of \(2^N\)

\(\cap \subseteq 2^N \times 2^N \times 2^N\) is defined as 

\[(A, B, C) \in \cap \iff A \cap B = C\]

and the following is true for all \(A, B, C \in 2^N\)

if \(A, B \in 2^N\) and \((A, B, C) \in \cap\) then \(C \in 2^N\)
Closure Property Fact 1

Since relations are sets, we can speak of one relation as being closed under one or more others. We show now the following:

CP Fact 1

Transitivity is a closure property.

Proof

Let $D$ be a set, let $Q$ be a ternary relation on $D \times D$, i.e. $Q \subseteq (D \times D)^3$ be such that

$$Q = \{(((a, b), (b, c), (a, c)) : a, b, c \in D\}$$

Observe that for any binary relation $R \subseteq D \times D$, $R$ is closed under $Q$ if and only if $R$ is transitive.
CP Fact1 Proof

The definition of closure of $R$ under $Q$ says: for any $x, y, z \in D \times D$,

$\text{if } x, y \in R \text{ and } (x, y, z) \in Q \text{ then } z \in R$

But $(x, y, z) \in Q \iff x = (a, b), y = (b, c), z = (a, c)$ and

$(a, b), (b, c) \in R \text{ implies } (a, c) \in R$

is a true statement for all $a, b, c \in D \iff R$ is transitive
Closure Property Fact 2

We show now the following

CP Fact 2

Reflexivity is a closure property

Proof

Let $D \neq \emptyset$, we define an unary relation $Q'$ on $D \times D$, i.e. $Q' \subseteq D \times D$ as follows

$$Q' = \{(a,a) : a \in D\}$$

Observe that for any $R$ binary relation on $D$, i.e. $R \subseteq D \times D$ we have that

$R$ is closed under $Q'$ if and only if $R$ is reflexive
Problem 6

Definition

Let \( D \) be a set, let \( n \geq 0 \) and let \( R \subseteq D^{n+1} \) be a \((n+1)\)-ary relation on \( D \). Then the subset \( B \) of \( D \) is said to be **closed under** \( R \) if \( b_{n+1} \in B \) whenever \((b_1, \ldots, b_n, b_{n+1}) \in R\).

Any property of the form "the set \( B \) is closed under relations \( R_1, R_2, \ldots, R_m \)" is called a **Closure Property** of \( B \).
CP Theorem

Prove the following Closure Property Theorem

**CP Theorem**

Let $P$ be a closure property defined by relations on a set $D$, and let $A \subseteq D$.

Then there is a unique minimal set $B$ such that $B \subseteq A$ and $B$ has property $P$.

**Proof** Consider the set of all subsets of $D$ that are closed under relations $R_1, R_2, \ldots, R_m$ and that have $A$ as a subset.

We call this set $S$. 

CP Theorem Proof

Consider now

\[ S = \{ X \in 2^D : A \subseteq X \text{ and } X \text{ is closed under } R_1, R_2, \ldots, R_m \} \]

We need to show that the poset \( S = (S, \subseteq) \) has a unique minimal element \( B \).

Observe that \( S \neq \emptyset \) as \( D \subseteq S \) and \( D \) is trivially closed under \( R_1, R_2, \ldots, R_m \) and by definition \( A \subseteq D \).

Consider then the set \( B \) which is the intersection of all sets in \( S \), i.e.

\[ B = \bigcap S \]

Obviously \( A \subseteq B \) and we have to show now that \( B \) is closed under all \( R_i \).
CP Theorem Proof

Suppose that $a_1, a_2, \ldots a_{n-1} \in B$, and $a_1, a_2, \ldots a_{n-1}, a_n \in R_i$

Since $B$ is the intersection of all sets in $S$, we have that $a_1, a_2, \ldots a_{n-1} \in X$, for all $X \in S$

But all sets in $S$ are closed under all $R_i$, they also contain $a_n$

Therefore $a_n \in B$ and hence $B$ is closed under all $R_i$

Moreover, $B$ is minimal, because there can be no proper subset $C$ of $B$, such that $A \subseteq C$ and $C$ is closed under all $R_i$

Because then $C$ would be a member of $S$ and thus $C$ would include $B$
Closure Property Theorem

**CP Theorem**

Let $P$ be a closure property defined by relations on a set $D$, and let $A \subseteq D$

Then there is a **unique minimal** set $B$ such that $B \subseteq A$ and $B$ has property $P$
Two Definition of $R^*$ Revisited

Definition 1

$R^*$ is called a reflexive, transitive closure of $R$ iff $R \subseteq R^*$ and is $R^*$ is reflexive and transitive and is the smallest set with these properties.

Definition 2

Let $R$ be a binary relation on a set $A$.

The reflexive, transitive closure of $R$ is the relation

$$R^* = \{(a, b) \in A \times A : \text{there is a path from } a \text{ to } b \text{ in } R\}$$

Equivalency Theorem

$R^*$ of the Definition 2 is the same as $R^*$ of the Definition 1 and hence richly deserves its name reflexive, transitive closure of $R$.
Equivalency of Two Definition of $R^*$

**Proof**  Let

$$R^* = \{(a, b) \in A \times A : \text{ there is a path from } a \text{ to } b \text{ in } R\}$$

$R^*$ is **reflexive** for there is a trivial path (case n=1) from $a$ to $a$, for any $a \in A$

$R^*$ is **transitive** as for any $a, b, c \in A$

if there is a path from $a$ to $b$ and a path from $b$ to $c$, then there is a path from $a$ to $c$

Clearly $R \subseteq R^*$ because there is a path from $a$ to $b$ whenever $(a, b) \in R$
Equivalency of Two Definition of $R^*$

Consider a set $S$ of all binary relations on $A$ that contain $R$ and are reflexive and transitive, i.e.

$$S = \{ Q \subseteq A \times A : R \subseteq Q \text{ and } Q \text{ is reflexive and transitive} \}$$

We have just proved that $R^* \in S$

We prove now that $R^*$ is the smallest set in the poset $(S, \subseteq)$, i.e. that for any $Q \in S$ we have that $R^* \subseteq Q$
Equivalency of Two Definition of $R^*$

Assume that $(a, b) \in R^*$. By Definition 2 there is a path $a = a_1, \ldots, a_k = b$ from $a$ to $b$ and let $Q \in S$

We prove by Mathematical Induction over the length $k$ of the path from $a$ to $b$

**Base case:** $k=1$

We have that the path is $a = a_1 = b$, i.e. $(a, a) \in R^*$ and $(a, a) \in Q$ from reflexivity of $Q$

**Inductive Assumption:**

Assume that for any $(a, b) \in R^*$ such that there is a path of length $k$ from $a$ to $b$ we have that $(a, b) \in Q$
Equivalency of Two Definition of $R^*$

Inductive Step:
Let $(a, b) \in R^*$ be now such that there is a path of length $k+1$ from $a$ to $b$, i.e., there is a path $a = a_1, \ldots, a_k, a_{k+1} = b$

By inductive assumption $(a = a_1, a_k) \in Q$ and by definition of the path $(a_k, a_{k+1} = b) \in R$

But $R \subseteq Q$ hence $(a_k, a_{k+1} = b) \in Q$ and $(a, b) \in Q$ by transitivity

This **ends the proof** that Definition 2 of $R^*$ implies the Definition 1

The inverse implication follows from the previously proven fact that reflexivity and transitivity are closure properties