cse581
Computer Science Fundamentals: Theory

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TCB - LECTURE 1

TCB - THEORY OF COMPUTATION BASICS

DM and TCB

PART 3: Special types of Binary Relations

PART 4: Finite and Infinite Sets

PART 5: Some Fundamental Proof Techniques

Theory of Computation BASICS

PART 6: Closures and Algorithms

PART 7: Alphabets and languages

PART 8: Finite Representation of Languages

Theory of Computation Basics

PART 6: Closures and Algorithms

Closures - Intuitive

Idea

Natural numbers N are **closed** under $+$, i.e. for given two natural numbers n, m we always have that $n + m \in N$

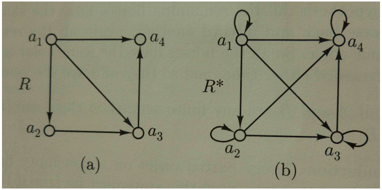
Natural numbers N are **not closed** under subtraction $-$, i.e there are two natural numbers n, m such that $n - m \notin N$, for example $1, 2 \in N$ and $1 - 2 \notin N$

Integers Z are **closed** under $-$, moreover Z is the smallest set containing N and closed under subtraction $-$

The set Z is called a **closure** of N under subtraction $-$

Closures - Intuitive

Consider the two directed graphs R (a) and R^* (b) as shown below



Observe that $R^* = R \cup \{(a_i, a_i) : \; i = 1, 2, 3, 4\} \cup \{(a_2, a_4)\}$,

$R \subseteq R^*$ and is R^* is reflexive and transitive whereas R is neither, moreover R^* is also the smallest set containing R that is reflexive and transitive

We call such relation R^* the reflexive, transitive closure of R

We define this concept formally in two ways and prove the equivalence of the two definitions

Two Definitions of R^*

Definition 1 of R^*

R^* is called a reflexive, transitive closure of R iff $R \subseteq R^*$ and is R^* is reflexive and transitive and is the smallest set with these properties

This definition is based on a notion of a **closure property** which is any property of the form " the set B is closed under relations $R_1, R_2, \ldots, R_m$"

We define it formally and prove that reflexivity and transitivity are closures properties

Hence we **justify** the name: reflexive, transitive closure of R for R^*

Two Definitions of R^*

Definition 2 of R^*

Let R be a binary relation on a set A

The **reflexive, transitive closure of** R is the relation

$$R^* = \{(a,b) \in A \times A : \quad \text{there is a path from a to b in R}\}$$

This is a much simpler definition- and algorithmically more interesting as it uses a simple notion of a path

We hence start our investigations from it- and only later introduce all notions needed for the **Definition 1** in order to prove that the R^* defined above is really what its name says: the **reflexive, transitive closure of** R

Definition 2 of R^*

We bring back the following

Path Definition

A **path** in the binary relation R is a finite sequence

$a_1, \ldots, a_n$ such that $(a_i, a_{i+1}) \in R$, for $i = 1, 2, \ldots n-1$ and $n \geq 1$

The path $a_1, \ldots, a_n$ is said to be from a_1 to a_n

The path a_1 (case when $n = 1$) always exist and is called a trivial path from a_1 to a_1

Definition 2

Let R be a binary relation on a set A

The **reflexive, transitive closure of** R is the relation

$$R^* = \{(a, b) \in A \times A : \text{ there is a path from a to b in } R\}$$

Algorithms

Definition 2 immediately suggests an following algorithm for computing the reflexive transitive closure R^* of any given binary relation R over some finite set $A = \{a_1, a_2, \ldots, a_n\}$

Algorithm 1

Initially $R^* := 0$

for $i = 1, 2, \ldots, n$ do

for each i- tuple $(b_1, \ldots, b_i) \in A^i$ do

if $b_1, \ldots, b_i$ is a **path in** R then add (b_1, b_n) to R^*

Algorithms

We also have a following much faster algorithm

Algorithm 2

Initially $R^* := R \cup \{(a_i, a_i) : \ a_i \in A\}$

for $j = 1, 2, \ldots, n$ do

for $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n$ do

if $(a_i, a_j), (a_j, a_k) \in R^*$ but $(a_i, a_k) \notin R^*$

then add (a_i, a_k) to R^*

Closure Property Formal

We introduce now formally a concept of a closure property of a given set

Definition

Let D be a set, let $n \geq 0$ and

let $R \subseteq D^{n+1}$ be a $(n+1)$-ary relation on D

Then the subset B of D is said to be **closed under** R

if $b_{n+1} \in B$ whenever $(b_1, \ldots, b_n, b_{n+1}) \in R$

Any property of the form " the set B is closed under relations $R_1, R_2, \ldots, R_m$" is called a **closure property** of B

Closure Property Examples

Observe that any function $f: D^n \longrightarrow D$ is a special relation $f \subseteq D^{n+1}$ so we have also defined what does it mean that a set $A \subseteq D$ is **closed under** the function f

E1: $+$ is a closure property of N

Adition is a function $+: N \times N \longrightarrow N$ defined by a formula $+(n, m) = n + m$, i.e. it is a **relation** $+ \subseteq N \times N \times N$ such that

$$+ = \{(n, m, n+m) : \quad n, m \in N\}$$

Obviously the set $N \subseteq N$ is (formally) closed under +
because

for any $n, m \in N$ we have that $(n, m, n+m) \in +$

Closures Property Examples

E2: $\cap$ is a closure property of 2^N

$\cap \subseteq 2^N \times 2^N \times 2^N$ is defined as

$$(A, B, C) \in \cap \quad \text{iff} \quad A \cap B = C$$

and the following is true for all $A, B, C \in 2^N$

$$\text{if } A, B \in 2^N \text{ and } (A, B, C) \in \cap \text{ then } C \in 2^N$$

Closure Property Fact1

Since relations are sets, we can speak of one relation as being closed under one or more others

We show now the following

CP Fact 1

Transitivity is a closure property

Proof

Let D be a set, let Q be a ternary relation on $D \times D$, i.e. $Q \subseteq (D \times D)^3$ be such that

$$Q = \{((a, b), (b, c), (a, c)) : \quad a, b, c \in D\}$$

Observe that for any binary relation $R \subseteq D \times D$,

R is closed under Q if and only if R is **transitive**

CP Fact1 Proof

The definition of closure of R under Q says: for any $x, y, z \in D \times D$,

$$\text{if } x, y \in R \text{ and } (x, y, z) \in Q \text{ then } z \in R$$

But $(x, y, z) \in Q$ iff $x = (a, b), y = (b, c), z = (a, c)$ and

$$(a, b), (b, c) \in R \text{ implies } (a, c) \in R$$

is a true statement for all $a, b, c \in D$ iff R is **transitive**

Closure Property Fact2

We show now the following

CP Fact 2

Reflexivity is a closure property

Proof

Let $D \neq \emptyset$, we define an **unary** relation Q' on $D \times D$, i.e. $Q' \subseteq D \times D$ as follows

$$Q' = \{(a, a) : \quad a \in D\}$$

Observe that for any R binary relation on D, i.e. $R \subseteq D \times D$ we have that

R is closed under Q' if and only if R is reflexive

Problem 6

Definition

Let D be a set, let $n \geq 0$ and

let $R \subseteq D^{n+1}$ be a $(n+1)$-ary relation on D

Then the subset B of D is said to be **closed under** R

if $b_{n+1} \in B$ whenever $(b_1, \ldots, b_n, b_{n+1}) \in R$

Any property of the form " the set B is closed under relations $R_1, R_2, \ldots, R_m$" is called a **Closure Property** of B

CP Theorem

Prove the following **Closure Property Theorem**

CP Theorem

Let P be a closure property defined by relations on a set D, and let $A \subseteq D$

Then there is a **unique minimal** set B such that $B \subseteq A$ and B has property P

Proof Consider the set if all subsets of D that are **closed** under relations $R_1, R_2, \ldots, R_m$ and that have A as a subset

We call this set $\mathcal{S}$

CP Theorem Proof

Consider now

$$\mathcal{S} = \{X \in 2^D : \ A \subseteq X \text{ and } X \text{ is closed under } R_1, R_2, \ldots, R_m\}$$

We need to show that the poset $\mathbf{S} = (\mathcal{S}, \subseteq)$ has a **unique minimal** element B.

Observe that $\mathcal{S} \neq \emptyset$ as $D \subseteq \mathcal{S}$ and D is trivially closed under $R_1, R_2, \ldots, R_m$ and by definition $A \subseteq D$.

Consider then the set B which is the intersection of all sets in $\mathcal{S}$, i.e.

$$B = \bigcap \mathcal{S}$$

Obviously $A \subseteq B$ and we have to show now that B is closed under all R_i

CP Theorem Proof

Suppose that $a_1, a_2, \ldots a_{n-1} \in B$, and

$a_1, a_2, \ \ldots \ a_{n-1}, \ a_n \in R_i$

Since B is the intersection of all sets in $\mathcal{S}$, we have that

$a_1, a_2, \ldots a_{n-1} \in X$, for all $X \in \mathcal{S}$

But all sets in $\mathcal{S}$ are closed under all R_i, they also contain a_n

Therefore $a_n \in B$ and hence B is closed under all R_i

Moreover, B is **minimal**, because there can be **no proper** subset C of B, such that $A \subseteq C$ and C is closed under all R_i

Because then C would be a member of $\mathcal{S}$ and thus C would include B

Closure Property Theorem

CP Theorem

Let P be a closure property defined by relations on a set D, and let $A \subseteq D$

Then there is a **unique minimal** set B such that $B \subseteq A$ and B has property P

Two Definition of R^* Revisited

Definition 1

R^* is called a reflexive, transitive closure of R iff $R \subseteq R^*$ and is R^* is reflexive and transitive and is the smallest set with these properties

Definition 2

Let R be a binary relation on a set A

The **reflexive, transitive closure of** R is the relation

$$R^* = \{(a,b) \in A \times A : \text{ there is a path from } a \text{ to } b \text{ in } R\}$$

EquivalencyTheorem

R^* of the **Definition 2** is the same as R^* of the **Definition 1** and hence richly deserves its name reflexive, transitive closure of R

Equivalency of Two Definition of R^*

Proof Let

$$R^* = \{(a, b) \in A \times A : \quad \text{there is a path from a to b in R}\}$$

R^* is reflexive for there is a trivial path (case n=1) from a to a, for any $a \in A$

R^* is transitive as for any $a, b, c \in A$

if there is a path from a to b and a path from b to c, then there is a path from a to c

Clearly $R \subseteq R^*$ because there is a path from a to b whenever $(a, b) \in R$

Equivalency of Two Definition of R^*

Consider a set $\mathcal{S}$ of all binary relations on A that contain R and are reflexive and transitive, i.e.

$$\mathcal{S} = \{Q \subseteq A \times A : \ R \subseteq Q \text{ and } Q \text{ is reflexive and transitive } \}$$

We have just proved that $R^* \in \mathcal{S}$

We prove now that R^* is the smallest set in the poset $(\mathcal{S}, \subseteq)$, i.e. that for any $Q \in \mathcal{S}$ we have that $R^* \subseteq Q$

Equivalency of Two Definition of R^*

Assume that $(a, b) \in R^*$. By Definition 2 there is a path $a = a_1, \dots, a_k = b$ from a to b and let $Q \in \mathcal{S}$

We prove by Mathematical Induction over the length k of the path from a to b

Base case: k=1

We have that the path is $a = a_1 = b$, i.e. $(a, a) \in R^*$ and $(a, a) \in Q$ from reflexivity of Q

Inductive Assumption:

Assume that for any $(a, b) \in R^*$ such that there is a path of length k from a to b we have that $(a, b) \in Q$

Equivalency of Two Definition of R^*

Inductive Step:

Let $(a, b) \in R^*$ be now such that there is a path of length k+1 from a to b , i.e there is a a path $a = a_1, \ldots, a_k, a_{k+1} = b$

By inductive assumption $(a = a_1, a_k) \in Q$ and by definition of the path $(a_k, a_{k+1} = b) \in R$

But $R \subseteq Q$ hence $(a_k, a_{k+1} = b) \in Q$ and $(a, b) \in Q$ by transitivity

This **ends the proof** that Definition 2 of R^* implies the Definition1

The inverse implication follows from the previously proven fact that reflexivity and transitivity are closure properties