ONE PROBLEM PART 1 (1pts)

Let \( L = L_{\{\neg, \sim, \Rightarrow, \rightarrow\}} \) be a language with one argument connectives \( \neg, \sim \) called strong negation and weak negation, and with two arguments connectives \( \Rightarrow, \rightarrow \) called strong implication and weak implication.

We define a 3 valued extensional semantics \( M \) for the language \( L_{\{\neg, \sim, \Rightarrow, \rightarrow\}} \) by defining the connectives \( \neg, \sim, \Rightarrow, \rightarrow \) as functions on the set \( \{F, \bot, T\} \) of 3 logical values as follows.

The functions \( \neg, \Rightarrow \) restricted to the set \( \{F, T\} \) are the same as in the classical case.

We extend them to the full set \( \{F, \bot, T\} \) for strong negation as \( \neg \bot = F \), and for strong implication as \( x \Rightarrow \bot = F \) for \( x = T, F \) and \( \bot \Rightarrow y = \begin{cases} \bot & \text{if } y = \bot \\ T & \text{otherwise} \end{cases} \) for \( x \in \{T, F\} \).

We define the weak negation \( \sim \) : \( \{T, \bot, F\} \to \{T, \bot, F\} \) as \( \sim x = \begin{cases} T & \text{if } x = \bot \\ x & \text{for } x \in \{T, F\} \end{cases} \)

The weak implication \( \rightarrow \) : \( \{T, \bot, F\} \times \{T, \bot, F\} \to \{T, \bot, F\} \) is defined for all \( x, y \in \{T, \bot, F\} \) as \( x \rightarrow y = \sim (x \Rightarrow y) \)

(1pts) Fill in the connectives tables. Remember that the \( M \) connectives \( \neg, \Rightarrow \) on set \( \{F, T\} \) are the same as classical \( \neg, \Rightarrow \).

\[
\begin{array}{c|ccc}
\neg & F & \bot & T \\
\hline
T & F & F & F
\end{array}
\quad
\begin{array}{c|ccc}
\sim & F & \bot & T \\
\hline
T & F & T & T
\end{array}
\quad
\begin{array}{c|ccc}
\Rightarrow & F & \bot & T \\
\hline
F & T & F & T \\
\bot & T & \bot & T \\
T & F & T & T
\end{array}
\quad
\begin{array}{c|ccc}
\rightarrow & F & \bot & T \\
\hline
F & T & F & T \\
\bot & T & T & T \\
T & F & T & T
\end{array}
\]

ONE PROBLEM PART 2 (1pts) Use shorthand notation.

(0.2pts) Prove that any truth assignment \( v \) such that \( v(a) = v(b) = \bot \) is a \( M \) model for the formula \( (\neg a \rightarrow (\sim a \Rightarrow \neg b)) \).

Solution We evaluate our formula for. \( a = b = \bot \) i.e. we evaluate \( v'((\neg a \rightarrow (\sim a \Rightarrow \neg b))) = \neg \bot \rightarrow (\neg \bot \Rightarrow \neg \bot)) = F \rightarrow (T \Rightarrow F) = F \rightarrow F = T \).

We write it symbolically \( v \models M (\neg a \rightarrow (\sim a \Rightarrow \neg b)) \).

This is not the only model. Find, as an exercise other models.

(0.3pts) \( \not\models_M (a \Rightarrow a) \) and \( \models_M (a \Rightarrow \sim \neg a) \).

Solution To prove \( \not\models_M (a \Rightarrow a) \) we have to find a counter MODEL \( v \) for \( (a \Rightarrow \sim \neg a) \).

Consider any \( v : VAR \to \{F, \bot, T\} \) such that \( v(a) = \bot \).

We evaluate \( \bot \Rightarrow \bot = F \) and so \( (a \Rightarrow a) \) is not a \( M \) tautology.
To prove that $\models_M (a \Rightarrow \neg \neg a)$ we first observe that it is a classical tautology and the $M$ connectives $\neg$, $\Rightarrow$ on set $\{F, T\}$ are the same as classical $\neg$, $\Rightarrow$, so to prove $\models_M (a \Rightarrow \neg \neg a)$ we have to consider only the case $a = \bot$

and get $\bot \Rightarrow \neg \neg \bot = \bot \Rightarrow \neg F = \bot \Rightarrow T = T$.

This ends the proof.

(0.2pts) Let $T$ be a set of classical tautologies, $LT$ be a set of Lukasiewicz tautologies, and $MT$ be a set of all $M$ tautologies. Prove that $T \cap MT \neq \emptyset$ and $LT \neq MT$.

Solution We just proved that the formula $(a \Rightarrow \neg \neg a) \in T \cap MT$, hence $T \cap MT \neq \emptyset$.

As we have proved that $\not\models_M (a \Rightarrow a)$, and we know that $(a \Rightarrow a) \in LT$, we proved that $LT \neq MT$.

(0.3pts) Prove that the semantics $M$ is well defined.

Solution By definition, semantics $M$ is well defined if and only if $MT \neq \emptyset$.

This is true as we have already proved that $(a \Rightarrow \neg \neg a) \in MT$.