ONE PROBLEM (2pts)

PART 1  Here is a mathematical statement $S$:

For each integer $m \in \mathbb{Z}$ the following holds: If $m > 5$, then there is a natural number $n \in \mathbb{N}$, such that $m + n > 5$

Re-write $S$ as a symbolic mathematical statement $SM$ that only uses mathematical and logical symbols.

Solution  $S$ becomes a symbolic mathematical statement $SM$:

$$SM : \forall m \in \mathbb{Z} (m > 5 \Rightarrow \exists n \in \mathbb{N} \ m + n > 5)$$

Translate the mathematical statement $SM$ into formula with restricted quantifiers of a to a corresponding predicate language $\mathcal{L}$. Explain your choice of symbols.

Solution  We write $Z(x)$ for $x \in \mathbb{Z}$, $N(y)$ for $y \in \mathbb{N}$, a constant $c$ for the number 5. We use $G \in \mathbb{P}$ to denote the relation $>$, we use $f \in \mathbb{F}$ to denote the function $+$. The statement $m > 5$ becomes an atomic formula $G(x, c)$. The statement $m + n > 5$ becomes an atomic formula $G(f(x, y), c)$.

The symbolic mathematical statement $SM$ becomes a restricted quantifiers formula

$$\forall Z(x) (x) (G(x, c) \Rightarrow \exists N(y) G(f(x, y), c))$$

Translate your restricted domain quantifiers formula into a correct formula $A$ of the predicate language $\mathcal{L}$.

Solution  We apply now the transformation rules and get a corresponding formula $A \in \mathcal{F}$:

$$\forall x (Z(x) \Rightarrow (G(x, c) \Rightarrow \exists y (N(y) \cap G(f(x, y), c))))$$

PART 2  Given a formula $A : \forall x \exists y P(f(x, y), c)$ of the predicate language $\mathcal{L}$, and two model structures $M_1 = (\mathbb{Z}, I_1)$, and $M_2 = (\mathbb{N}, I_2)$

with the interpretations defined as follows.

$$P_{I_1} : = \cdot, \ f_{I_1} : +, \ c_{I_1} : 0 \ \text{and} \ P_{I_2} : >, \ f_{I_2} : \cdot, \ c_{I_2} : 0$$

Show that $M_1 \models A$

Solution  $M_1 \models A$ because $A_{I_1} : \forall x \exists y Z(x + y = 0$ is a true mathematical statement as we have that each $x \in \mathbb{Z}$ exists $y = -x$ and $-x \in \mathbb{Z}$ and $x - x = 0$

Show that $M_2 \not\models A$

Solution  $M_2 \not\models A$ because $A_{I_2} : \forall x \exists y \in \mathbb{N} x \cdot y > 0$ is a false statement for $x = 0$. 
