MIDTERM has three parts: Part 1: 6 Problems, total (90pts), Part 2: 5 Yes/No DM Definitions, total (10pts) and Part 3: 5 Yes/No DM Questions, total (10pts). Extra Credit 10pts is included in the Total sum of 110 pts for the test.

PART 1: PROBLEMS

Problem 1 (20 pts)

Write the following natural language statement:

*From the fact that there is a bird that does not fly and 4 + 4 = 4, we deduce the following: it is not possible that all birds fly OR it is not necessary that 4 + 4 = 4.*

in the following 2 WAYS:

1. (10 pts) As a formula \( A_1 \in \mathcal{F}_1 \) of a language \( \mathcal{L}_{\{\neg, \Box, \lozenge, \cap, \cup, \Rightarrow\}} \). Use Propositional Variables \( a, b, c \) for consecutive statements.

   **Solution**

   We use Propositional Variables \( a, b, c \) where

   - \( a \) denotes statement: *there is a bird that does not fly*
   - \( b \) denotes statement: \( 4 + 4 = 4 \)
   - \( c \) denotes statement: *all birds fly*

   The formula \( A_1 \in \mathcal{F}_1 \) is:

   \[
   (a \land b) \Rightarrow (\neg c \lor \neg b)
   \]

2. (10pts) As a formula \( A_2 \in \mathcal{F}_2 \) of a PREDICATE LANGUAGE language \( \mathcal{L}(P, F, V) \) with the set \( \{\neg, \Box, \lozenge, \cap, \cup, \Rightarrow\} \) of propositional connectives.

   Use the following Predicates, Functions and Constants:

   - \( B(x) \) for \( x \) is a bird, \( F(x) \) for \( x \) can fly, \( E(x, y) \) for \( x = y \), \( f(x, y) \) for +, and \( c \) for 4.

   **Solution**

   Restricted domain formula is:

   \[
   ((\exists x B(x) \land \neg F(x)) \land E(f(c, c), c)) \Rightarrow (\neg \forall x (B(x) \Rightarrow F(x)) \cup \neg \Box E(f(c, c), c))
   \]

   Formula \( A_2 \in \mathcal{F}_2 \) is:

   \[
   ((\exists x (B(x) \land \neg F(x)) \land E(f(c, c), c)) \Rightarrow (\neg \forall x (B(x) \Rightarrow F(x)) \cup \neg \Box E(f(c, c), c)))
   \]

Problem 2 (10 pts)

Given a formula \( A : \forall x \exists y P(f(x, y), c) \) of the predicate language \( \mathcal{L} \), and two model structures \( M_1 = (N, I_1) \), \( M_2 = (N, I_2) \) with the interpretations defined as follows.

- \( P_{I_1} := , \quad f_{I_1} := +, \quad c_{I_1} := 0 \) and \( P_{I_2} := , \quad f_{I_2} := \cdot, \quad c_{I_2} := 0 \).

   **Verify** whether \( M_1 \models A \) and \( M_2 \not\models A \)

   **Solution**

   \( M_1 \not\models A \) because \( A_{I_1} : \forall x \exists y \exists z : x + y = 0 \) is a false mathematical statement as we have that each natural number \( x > 0 \) does not \( y \in N \) such that \( x + y = 0 \).
Show that \( M_2 \not\models A \)

**Solution**

\( M_2 \not\models A \) because \( A_{I_2} : \forall x \in \mathbb{N} \exists y \in \mathbb{N} x \cdot y > 0 \) is a false statement for \( x = 0 \).

**Problem 3 (10 pts)**

1. **Circle** formulas that are propositional tautologies  
   \[ S_1 = \{ ((\neg c \land c) \Rightarrow (\neg b \Rightarrow (d \land e))), ((a \Rightarrow b) \lor (a \Rightarrow b)), ((a \land \neg b) \Rightarrow ((a \land \neg b) \Rightarrow (\neg d \lor e))), (\neg a \Rightarrow (\neg a \lor b)) \} \]
   
   **Solution** \( \not\models ((a \land \neg b) \Rightarrow ((a \land \neg b) \Rightarrow (\neg d \lor e))), \) all other formulas are tautologies

2. **Circle** formulas that are predicate tautologies  
   \[ S_2 = \{ (\exists x A(x) \Rightarrow \forall x \neg A(x)), (\forall x (P(x, y) \land Q(y)) \Rightarrow \neg \exists x \neg (P(x, y) \land Q(y)),
   ((\exists x A(x) \land \exists x B(x)) \Rightarrow \exists x (A(x) \land B(x))), (\forall x (A(x) \Rightarrow B) \Rightarrow (\exists x A(x) \Rightarrow B)) \} \]
   
   **Solution** \( \not\models ((\exists x A(x) \land \exists x B(x)) \Rightarrow \exists x (A(x) \land B(x))), \) all other formulas are tautologies

**Problem 4 (15 pts)**

We define a 3 valued extensional semantics \( M \) for the language \( L_{\{\neg, \lor, \Rightarrow\}} \) by defining the connectives \( \neg, \lor, \Rightarrow \) on a set \( \{F, \bot, T\} \) of logical values by the following truth tables.

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1. (5pts)

Verify whether a formula \( A = (a \lor (b \Rightarrow (\neg a \lor c))) \) has a model/counter model under the semantics \( M \). You can use shorthand notation.

**Solution**

Observe that the main connective of \( A \) is a disjunction, hence any \( v \), such that \( v(a) = T \) is a \( M \)-model for \( A \). We evaluate: \( (T \lor (b \Rightarrow (\neg T \lor c))) = T \) for any logical values for \( b \) and \( c \).

Observe now that any \( v \) to be a \( M \)-counter model for \( A \) must be such that \( v(a) = T \) and \( v^\ast((b \Rightarrow (\neg a \lor c))) = F \).

We evaluate: \( (b \Rightarrow (\neg a \lor c)) = (b \Rightarrow (\neg F \lor c)) = (b \Rightarrow (T \lor c) = (b \Rightarrow T) \) for any values for \( c \).
But \( (b \Rightarrow T) = T \) for any logical value of \( b \). This proves that a **M-counter model** for \( A \) does not exist.

This means that the formula \( A \) is an **M-tautology**, i.e., we proved

\[ \models_M (a \cup (b \Rightarrow (\neg a \cup c))) \]

2. **(10pts)** Verify whether the following set \( G \) is **M-consistent**. You can use shorthand notation

\[ G = \{ La, (a \cup \neg La), (a \Rightarrow b), b \} \]

**Solution**

Any \( \nu \), such that \( \nu(a) = T, \nu(b) = T \) is a **M model** for \( G \) as

\[ LT = T, (T \cup \neg LT) = T, (T \Rightarrow T) = T, b = T. \]

This may not be the only model.

**Problem 5 (20 pts)**

Let \( S \) be the following proof system

\[ S = (L_{\neg,\cup,\Rightarrow}, \mathcal{F}, \{ A_1, A_2 \}, \{ r_1, r_2 \}) \]

for the logical axioms and rules of inference defined for any formulas \( A, B \in \mathcal{F} \) as follows

**Logical Axioms**

\[ A_1 (La \cup \neg La) \]
\[ A_2 (A \Rightarrow La) \]

**Rules** of inference:

\[ (r_1) \frac{A; B}{(A \cup B)} \quad (r_2) \frac{A}{L(A \Rightarrow B)} \]

1. **(5pts)** We know by an easy evaluation that the axioms \( A_1, A_2 \) are **M-tautologies** under semantics \( M \) defined in **Problem 4** above.

**Verify** whether the system \( S \) is **M-sound**

You can use shorthand notation

**Solution**

We know that both logical axioms of \( S \) are **M-tautologies**

So we only have to check the **M soundness** of the inference rules of \( S \)

Rule (r1) is **M-sound** because when \( A = T \) and \( B = T \) we get \( A \cup B = T \cup T = T. \)

Rule (r2) is **not M-sound** because when \( A = T \) and \( B = F \) (or \( B = \bot \)) we get \( L(A \Rightarrow B) = L(T \Rightarrow F) = LF = F \) or \( L(T \Rightarrow \bot) = L \bot = F \)

We proved that \( S \) is **not M-sound**.

2. **(10pts)** Show, by constructing a proper **formal proof** that

\[ \vdash_S ((Lb \cup \neg Lb) \cup L((La \cup \neg La) \Rightarrow b))) \]
You must write comments how each step pot the proof was obtained

**Solution**

Here is the proof \( B_1, B_2, B_3, B_4 \)

\[ B_1: \quad (La \lor \neg La) \quad \text{Axiom} \quad A_1 \quad \text{for} \quad A = a \]

\[ B_2: \quad L((La \lor \neg La) \Rightarrow b) \quad \text{rule} \quad r2 \quad \text{for} \quad B=b \quad \text{applied to} \quad B_1 \]

\[ B_3: \quad (Lb \lor \neg Lb) \quad \text{Axiom} \quad A_1 \quad \text{for} \quad A=b \]

\[ B_4: \quad (Lrb \lor \neg Lrb) \quad \text{Axiom} \quad A_1 \quad \text{for} \quad A=b \]

**PROBLEM 6 (15pts)**

1. (5pts) Let \( \approx \) be an equivalence relation on a set \( A \neq \emptyset \).
   **Prove** that each equivalence class of \( \approx \) is non-empty.

   **Solution** This holds as \( a \in [a] \) for all \( a \in A \) by the reflexivity of the equivalence relation.

2. (10pts) Let \( \approx \) be an equivalence on the set \( N \) of natural numbers as follows. For any \( n, m \in N \),

\[ n \approx m \quad \text{if and only if} \quad (n)_3 = (m)_3 \]

where \((n)_3\) denotes a remainder of division \( n \) by 3.

   **Show** that \([7] = [1], \quad [9] = [0] \) and \([8] = [2]\) (5pts).

   **Solution** We evaluate using the definition of the equivalence class

\[ [7] = \{ m \in N : \quad 7 \approx m \} = \{ m \in N : \quad (7)_3 = (m)_3 \} = \{ m \in N : \quad 1 = (m)_3 \} \]

\[ [7] = \{ m \in N : \quad m = 3k + 1 \quad \text{for} \quad k \in N \} = \{ 1, 4, 7, 10, \ldots \ldots \} = [1] = [4] = [7] = [10] = \ldots \ldots \]

\[ [9] = \{ m \in N : \quad 9 \approx m \} = \{ m \in N : \quad (9)_3 = (m)_3 \} = \{ m \in N : \quad 0 = (m)_3 \} \]

\[ [9] = \{ m \in N : \quad m = 3k \quad \text{for} \quad k \in N \} = \{ 0, 3, 9, 12, \ldots \ldots \} = [0] = [3] = [9] = [12] = \ldots \ldots \]

\[ [8] = \{ m \in N : \quad 8 \approx m \} = \{ m \in N : \quad (8)_3 = (m)_3 \} = \{ m \in N : \quad 2 = (m)_3 \} \]

\[ [8] = \{ m \in N : \quad m = 3k + 2 \quad \text{for} \quad k \in N \} = \{ 2, 4, 7, 10, \ldots \ldots \} = [1] = [4] = [7] = [10] = \ldots \ldots \]

**List all elements** of the PARTITION \( P = N/ \approx \) (5pts)

**Solution**

Remember that any element of an equivalence class can be chosen as its reprezentant.

For example, we can and often choose in this case 0, 1, 2 \( \in N \) - the reminders of a division of \( n \in N \) by 3

and write

\[ P = N/ \approx = \{ [0], [1], [2] \} = \{ E_0, E_1, E_2 \} \]

where

\[ E_0 = \{ m \in N : \quad m = 3k \quad \text{for} \quad k \in N \} \]

\[ E_1 = \{ m \in N : \quad m = 3k + 1 \quad \text{for} \quad k \in N \} \]

\[ E_1 = \{ m \in N : \quad m = 3k + 2 \quad \text{for} \quad k \in N \} \]
PART 2 : Yes/No DM Definitions (10 pts)

Here are 5 definitions with provided Yes/No answers about their correctness. Some of provided answers are wrong. Read carefully. Circle only the provided answer that is correct.

1. onto function $f : A \rightarrow B$ if and only if $\forall b \in B \exists a \in A \ f(a) = b$  

2. inverse image $x \in f^{-1}(B)$ if and only if $f(x) \in B$, for any $f : X \rightarrow Y$ and any $B \subseteq Y$  

3. Generalized Intersection Given a sequence $\{A_n\}_{n \in \mathbb{N}}$ of sets, $\bigcap_{n \in \mathbb{N}} A_n = \{x : \exists n \in \mathbb{N} \ x \in A_n\}$  

4. Partition A family of sets $P \subseteq \mathcal{P}(A)$ is called a partition of the set $A$ if and only if

   1. $\forall X \in P \ (X \neq \emptyset)$,  
   2. $\forall X, Y \in P \ (X \cup Y = \emptyset)$,  
   3. $\bigcup P = A$  

5. Cardinality Aleph zero We say that a set $A$ has a cardinality $\aleph_0 (|A| = \aleph_0)$ if and only if $|A| = |\mathbb{N}|$  

PART 3 : Yes/No DM Questions (10 pts)

There are 5 Questions; 2 points each. Circle your answer. Write proper justification. No justification, no Credit.

1. $\{\{0, 1\}\} \in 2^{\{0,1\}, \{0, 1\}}$  

   Justify: Observe $2^{\{0,1\}, \{0, 1\}} = \{\emptyset, \{0\}, \{0, 1\}\} \ldots$  

2. Let $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Z}$ be given by a formula $f(n, m) = n - m^2$. $f$ is not $1-1$ function.  

   Justify: Take $(1, 0) \neq (2, 1)$. We get $f(1, 0) = 1 = f(2, 1)$. For example, we get $f(m^2, m) = m^2 - m^2 = 0$ for all $m \in \mathbb{N}$.  

3. Set $A$ is countable if and only if $|A| = |\mathbb{N}|$, where $\mathbb{N}$ is the set of natural numbers  

   Justify: Set $A$ is countable if and only if $A$ is FINITE or $|A| = |\mathbb{N}|$, where $\mathbb{N}$ is the set of natural numbers  

4. Let $f : \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})$ be given by formula: $f(n) = \{m \in \mathbb{N} : m < n^2\}$.  

   We have that $\emptyset \in f(\{0, 1, 2, 3\}$ and $0 \in f(\{0, 1, 2, 3\}$  

   Justify: $f(0) = \emptyset$, so $\emptyset \in f(\{0, 1, 2, 3\}$, but $0 \notin f(\{0, 1, 2, 3\}$  

   as $f(1) = \{m \in \mathbb{N} : m < 1^2\} = \{0\}$, and $\{0\} \in f(\{0, 1, 2, 3\}$  

5. Let $A_n = \{x \in \mathbb{R} : 0 < x < n\}$. The family $\{A_n\}_{n \in \mathbb{N}}$ form a partition of $\mathbb{R}$.  

   Justify: $\bigcup_{n \in \mathbb{N}} A_n \neq \mathbb{R}$