CSE581 Extra Credit Q2 SOLUTIONS Fall 2024
(4pts)

Please take your time and write **carefully** your solutions. There is no NO PARTIAL CREDIT.

QUESTION 1 We define a 3 valued extensional semantics **M** for the language $\mathcal{L}_{\{\neg, \mathbf{L}, \cup, \Rightarrow\}}$ by **defining the connectives** $\neg$, $\cup$, $\Rightarrow$ on a set $\{F, \perp, T\}$ of logical values by the following truth tables.

L	F	$\perp$	T
	F	*F*	T

$\neg$	F	$\perp$	T
	T	*F*	F

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	*T*	$\perp$	T
T	F	*F*	T

$\cup$	F	$\perp$	T
F	F	$\perp$	T
$\perp$	$\perp$	T	T
T	T	*T*	T

P1 0.5pts Verify whether the set $\mathbf{G} = \{\ \mathbf{L}a,\ (a \cup \neg\mathbf{L}b),\ (a \Rightarrow b),\ b, \neg\neg\mathbf{L}(c \cup b)\}$ is **M - consistent**.

Use shorthand notation.

Solution

Any v, such that $v(a) = T$, $v(b) = T$ is a **M model** for **G** as

$$\mathbf{L}T = T, \quad (T \cup \neg\mathbf{L}T) = T, \quad (T \Rightarrow T) = T, \quad b = T, \quad \neg\neg\mathbf{L}(c \cup T) = \neg\neg\mathbf{L}T = \neg F = T$$

Observe that **G** has 3 **restricted M models**: $a = T, b = T, c = T, c = \perp,\ c = F$

P2 0.5pts Let S be the following **proof system** $S = (\ \mathcal{L}_{\{\neg, \mathbf{L}, \cup, \Rightarrow\}},\ \mathcal{F},\ \{\mathbf{A1}, \mathbf{A2}\},\ \{r1,\ r2\}\ )$ where

A1 $(\mathbf{L}A \cup \neg\mathbf{L}A)$, **A2** $(A \Rightarrow \mathbf{L}A)$ and $(r1)\ \frac{A\ ;\ B}{(A \cup B)}$ $(r2)\ \frac{A}{\mathbf{L}(A \Rightarrow B)}$

Given a formula $A = (\mathbf{L}b \cup \neg\mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg\mathbf{L}a) \Rightarrow b)))$.

Prove that $\vdash_S\ (\mathbf{L}b \cup \neg\mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg\mathbf{L}a) \Rightarrow b)))$ by constructing its proper **formal proof** below.

Write **comments** how each step was obtained.

Solution

Here is the formal proof.

B_1: $(\mathbf{L}a \cup \neg\mathbf{L}a)$ Axiom A_1 for A= a

B_2: $\mathbf{L}((\mathbf{L}a \cup \neg\mathbf{L}a) \Rightarrow b)$ rule r2 for B=b applied to B_1

B_3: $(\mathbf{L}b \cup \neg\mathbf{L}Ab)$ Axiom A_1 for A=b

B_4: $((\mathbf{L}b \cup \neg\mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg\mathbf{L}a) \Rightarrow b))$ r1 applied to B_3 and B_2

P3 0.5pts Does the above $\vdash_S\ (\ \mathbf{L}b \cup \neg\mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg\mathbf{L}a) \Rightarrow b)))$ prove that $\models_{\mathbf{M}}\ (\mathbf{L}b \cup \neg\mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg\mathbf{L}a) \Rightarrow b)))$? Justify your answer.

Solution **No**, it doesn't because the system S is **not sound** as the rule $(r2)\ \frac{A}{\mathbf{L}(A \Rightarrow B)}$ is not sound.

Assume $A = T$, we get $\mathbf{L}(A \Rightarrow B) = \mathbf{L}(T \Rightarrow B) = F$, for any $B \neq T$.

QUESTION 2

Consider the Hilbert system $H1 = (\ \mathcal{L}_{\{\Rightarrow\}},\ \ \mathcal{F},\ \ \{A1, A2\},\ \ (MP)\ \frac{A\ ;\ (A\Rightarrow B)}{B}\ )$ where for any $A, B \in \mathcal{F}$
$A1:\ \ (A \Rightarrow (B \Rightarrow A)),\quad A2:\ \ ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$

P1 (0.5pts) Use the **Deduction Theorem** (we know that is holds for $H1$) to **prove** that

$$(\neg a \Rightarrow ((b \Rightarrow \neg a) \Rightarrow b)) \vdash_{H1} ((b \Rightarrow \neg a) \Rightarrow (\neg a \Rightarrow b))$$

Hint First apply Deduction Theorem twice.

Solution

$(\neg a \Rightarrow ((b \Rightarrow \neg a) \Rightarrow b)) \vdash_{H1} ((b \Rightarrow \neg a) \Rightarrow (\neg a \Rightarrow b))$ if and only if

$(\neg a \Rightarrow ((b \Rightarrow \neg a) \Rightarrow b)),\ (b \Rightarrow \neg a) \vdash_{H1} (\neg a \Rightarrow b))$ if and only if

$(\neg a \Rightarrow ((b \Rightarrow \neg a) \Rightarrow b)),\ (b \Rightarrow \neg a),\ \neg a \vdash_{H1} b$

We now construct a proof of $(\neg a \Rightarrow ((b \Rightarrow \neg a) \Rightarrow b)),\ (b \Rightarrow \neg a),\ \neg a \vdash_{H1} b$ as follows

$B_1:\ (\neg a \Rightarrow ((b \Rightarrow \neg a) \Rightarrow b))$ hypothesis

$B_2:\ (b \Rightarrow \neg a)$ hypothesis

$B_3:\ \neg a$ hypothesis

$B_4:\ ((b \Rightarrow \neg a) \Rightarrow b))$ B_1 , B_3 and (MP)

$B_5:\ b$ B_2, B_4 and (MP)

P2 (0.5pts) Let $H2$ be a **complete the proof system** obtained from the system $H1$ by **extending the language** to contain the negation $\neg$ and **adding** one additional axiom:

A3 $(B \Rightarrow ((\neg A \Rightarrow \neg B) \Rightarrow B))$

Let $H3$ be the proof system obtained from the system $H2$ by **adding** additional axiom

A4 $(B \Rightarrow ((\neg A \Rightarrow \neg B) \Rightarrow B))$

Does **Completeness Theorem** hold for $H3$? **Justify**.

Solution

Yes, $H3$ is a complete proof system, as the added axiom **A4** is a substitution of the axiom **A1** of $H2$.