cse581
COMPUTER SCIENCE FOUNDAMENTALS: THEORY

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Lecture 1

DISCRETE MATHEMATICS BASICS

Discrete Mathematics Basics

PART 1: Sets and Operations on Sets

PART 2: Relations and Functions

PART 3: Special types of Binary Relations

PART 4: Finite and Infinite Sets

PART 5: Some Fundamental Proof Techniques

PART 6: Closures and Algorithms

PART 7: Alphabets and languages

PART 8: Finite Representation of Languages

Discrete Mathematics Basics

PART 1: Sets and Operations on Sets

Sets

Set A set is a collection of objects

Elements The objects comprising a set are are called its elements or members

$a \in A$ denotes that a is an **element** of a set A

$a \notin A$ denotes that a is not an **element** of A

Empty Set is a set without elements

Empty Set is denoted by $\emptyset$

Sets

Sets can be defined by listing their elements;

Example

The set

$$A = \{a, \emptyset, \{a, \emptyset\}\}$$

has 3 elements:

$$a \in A, \quad \emptyset \in A, \quad \{a, \emptyset\} \in A$$

Sets

Sets can be defined by referring to other sets and to **properties** P(x) that elements may or may not have

We write it as

$$B = \{x \in A : \quad P(x)\}$$

Example

Let N be a set of natural numbers

$$B = \{n \in N : \quad n < 0\} = \emptyset$$

Operations on Sets

Set Inclusion

$A \subseteq B$ if and only if $\forall a(a \in A \Rightarrow a \in B)$ is a **true** statement

Set Equality

$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$

Proper Subset

$A \subset B$ if and only if $A \subseteq B$ and $A \neq B$

Operations on Sets

Subset Notations

$A \subseteq B$ for a subset (might be improper)

$A \subset B$ for a proper subset

Power Set Set of all subsets of a given set

$$\mathcal{P}(A) = \{B : \quad B \subseteq A\}$$

Other Notation

$$2^A = \{B : \quad B \subseteq A\}$$

Operations on Sets

Union

$A \cup B = \{x : \quad x \in A \quad \text{or} \quad x \in B\}$

We write:

$x \in A \cup B$ if and only if $x \in A \ \cup \ x \in B$

Intersection

$A \cap B = \{x : \quad x \in A \quad \text{and} \quad x \in B\}$

We write:

$x \in A \cap B$ if and only if $x \in A \cap \ x \in B$

Operations on Sets

Relative Complement

$x \in (A - B)$ if and only if $x \in A$ and $x \notin B$

We write:

$$A - B = \{x : \ x \in A \ \cap \ x \notin B\}$$

Complement is defined only for $A \subseteq U$, where U is called an **universe**

$$-A = U - A$$

We write for $x \in U$,

$x \in -A$ if and only if $x \notin A$

Operations on Sets

Algebra of sets consists of properties of sets that are **true** for all sets involved

We use **tautologies** of propositional logic

to prove **basic** properties of the algebra of sets

We then use the basic properties to **prove** more elaborated properties of sets

Operations on Sets

It is possible to form intersections and unions of **more** then two, or even a finite number o **sets**

Let $\mathcal{F}$ denote is any **collection** of sets

We write $\bigcup \mathcal{F}$ for the **set** whose **elements** are the elements of **all** of the sets in $\mathcal{F}$

Example Let

$$\mathcal{F} = \{\{a\}, \{\emptyset\}, \{a, \emptyset, b\}\}$$

We get

$$\bigcup \mathcal{F} = \{a, \ \emptyset, \ b\}$$

Operations on Sets

Observe that given

$$\mathcal{F} = \{\{a\}, \{\emptyset\}, \{a, \emptyset, b\}\} = \{A_1,\ A_2,\ A_3\}$$

we have that

$$\{a\} \cup \{\emptyset\} \cup \{a, \emptyset, b\} = A_1 \cup\ A_2 \cup A_3 = \{a,\ \emptyset,\ b\} = \bigcup \mathcal{F}$$

Hence we have that for any element x,

$x \in \bigcup \mathcal{F}$ if and only if there exists i, such that $x \in A_i$

Operations on Sets

We **define** formally

Generalized Union of any family $\mathcal{F}$ of sets is

$$\bigcup \mathcal{F} = \{x : \text{ exists a set } S \in \mathcal{F} \text{ such that } x \in S\}$$

We write it also as

$$x \in \bigcup \mathcal{F} \quad \text{if and only if} \quad \exists_{S \in \mathcal{F}} \; x \in S$$

Operations on Sets

Generalized Intersection of any family $\mathcal{F}$ of sets is

$$\bigcap \mathcal{F} = \{x : \quad \forall_{S \in \mathcal{F}}\ x \in S\}$$

We write

$$x \in \bigcap \mathcal{F} \quad \text{if and only if} \quad \forall_{S \in \mathcal{F}}\ x \in S$$

Operations on Sets

Ordered Pair

Given two sets A, B we denote by

$$(a, b)$$

an **ordered pair**, where $a \in A$ and $b \in B$

We call a a **first** coordinate of (a, b)

and b its **second** coordinate

We define

$$(a, b) = (c, d) \quad \text{if and only if} \quad a = c \quad \text{and} \quad b = d$$

Operations on Sets

Cartesian Product

Given two sets A and B, the set

$$A \times B = \{(a, b): \quad a \in A \text{ and } b \in B\}$$

is called a **Cartesian product** (cross product) of sets A, B

We write

$$(a, b) \in A \times B \quad \text{if and only if} \quad a \in A \text{ and } b \in B$$

Discrete Mathematics Basics

PART 2: Relations and Functions

Binary Relations

Binary Relation

Any set R such that $R \subseteq A \times A$

is called a **binary relation** defined in a set A

Domain, Range of R

Given a binary relation $R \subseteq A \times A$, the set

$$D_R = \{a \in A : \ (a, b) \in R\}$$

is called a **domain** of the relation R

The set

$$V_R = \{b \in A : \ (a, b) \in R\}$$

is called a **range** (set of values) of the relation R

n- ary Relations

Ordered tuple

Given sets $A_1, ...A_n$, an element $(a_1, a_2, ...a_n)$ such that $a_i \in A_i$ for $i = 1, 2, ...n$ is called an **ordered tuple**

Cartesian Product of sets $A_1, , A_n$ is a set

$$A_1 \times A_2 \times ... \times A_n = \{(a_1, a_2, ...a_n) : \; a_i \in A_i, i = 1, 2, ...n\}$$

n-ary Relation on sets $A_1, \ldots, A_n$ is any subset of $A_1 \times A_2 \times ... \times A_n$, i.e. the set

$$R \subseteq A_1 \times A_2 \times \ \ldots \ \times A_n$$

Binary Relations

Binary Relation

Any set R such that $R \subseteq A \times B$

is called a **binary relation** defined in a sets A and B

Domain, Range of R

Given a binary relation $R \subseteq A \times B$, the set

$$D_R = \{a \in A : \ (a, b) \in R\}$$

is called a **domain** of the relation R

The set

$$V_R = \{b \in B : \ (a, b) \in R\}$$

is called a **range** (set of values) of the relation R

Function as Relation

Definition

A binary relation $R \subseteq A \times B$ on sets A, B is a **function** from A to B

if and only if the following condition holds

$$\forall_{a \in A} \ \exists!_{\, b \in B} \ (a, b) \in R$$

where $\exists!_{\, b \in B}$ means there is **exactly one** $b \in B$

Because the condition says that for any $a \in A$ we have **exactly one** $b \in B$, we write

$$R(a) = b \quad \text{for} \quad (a, b) \in R$$

Function as Relation

Given a binary relation

$$R \subseteq A \times B$$

that is a **function**

The set A is called a **domain** of the function R and we write:

$$R : \quad A \longrightarrow B$$

to denote that the **relation** R is a **function** and say that R **maps** the set A **into** the set B

Functions

Function notation

We denote relations that are functions by letters f, g, h,...
and write

$$f: \quad A \longrightarrow B$$

say that the function f **maps** the set A **into** the set B

Domain, Codomain

Let $f: \quad A \longrightarrow B,$

the set A is called a **domain** of f ,

and the set B is called a **codomain** of f

Range

Given a function $f: \quad A \longrightarrow B$

The set

$$R_f = \{b \in B: \quad b = f(a) \text{ and } a \in A\}$$

is called a **range** of the function f

By definition, the **range** of f is a subset of its **codomain** B

We write $R_f = \{b \in B: \quad \exists_{a \in A}\ b = f(a)\}$

The set

$$f = \{(a, b) \in A \times B: \quad b = f(a)\}$$

is called a **graph** of the function f

Functions

Function *"onto"*

The function $f: \ A \longrightarrow B$ is an **onto** function if and only if the following condition holds

$$\forall_{b \in B} \, \exists_{a \in A} \, f(a) = b$$

We denote it by

$$f: \ A \xrightarrow{onto} B$$

Functions

Function *" one- to -one"*

The function $f: \; A \longrightarrow B$

is called a **one- to -one** function and denoted by

$$f: \; A \stackrel{1-1}{\longrightarrow} B$$

if and only if the following condition holds

$$\forall_{x,y \in A} (x \neq y \Rightarrow f(x) \neq f(y) \;)$$

Functions

A function $f: \ A \longrightarrow B$ is **not one- to -one** function if and only if the following condition holds

$$\exists_{x,y \in A}(x \neq y \cap f(x) = f(y)\ )$$

If a function f is **1-1** and **onto** we denote it as

$$f: \ A \overset{1-1, onto}{\longrightarrow} B$$

Functions

Composition of functions

Let f and g be two functions such that

$$f: \quad A \longrightarrow B \quad \text{and} \quad g: \quad B \longrightarrow C$$

We **define** a new function

$$h: \quad A \longrightarrow C$$

called a **composition** of functions f and g as follows:

for any $x \in A$ we put

$$h(x) = g(f(x))$$

Composition notation

Given function f and g such that

$$f: \quad A \longrightarrow B \quad \text{and} \quad g: \quad B \longrightarrow C$$

We **denote** the **composition** of f and g by $(f \circ g)$ in order to stress that the function

$$f: \quad A \longrightarrow \mathbf{B}$$

"goes first" followed by the function

$$g: \quad \mathbf{B} \longrightarrow C$$

with a **shared** set **B** between them

Functions

We write now the **definition** of composition of functions f and g using the **composition notation** (name for the composition function) $(f \circ g)$ as follows

The composition $(f \circ g)$ is a **new** function

$$(f \circ g): \quad A \longrightarrow C$$

such that for any $x \in A$ we put

$$(f \circ g)(x) = g(f(x))$$

Functions

There is also other notation (name) for the **composition** of f and g that uses the symbol $(g \circ f)$, i.e. we put

$$(g \circ f)(x) = g(f(x)) \quad \text{for all } x \in A$$

This notation was invented to help calculus students to remember the formula g(f(x)) defining the composition of functions f and g

Functions

Inverse function

Let $f: A \longrightarrow B$ and $g: B \longrightarrow A$

g is called an **inverse** function to f if and only if the following condition holds

$$\forall_{a \in A} (f \circ g)(a) = g(f(a)) = a$$

If g is an **inverse** function to f we denote by $g = f^{-1}$

Functions

Identity function

A function $I: \quad A \longrightarrow A$ is called an **identity** on A

if and only if the following condition holds

$$\forall_{a \in A} I(a) = a$$

Inverse and Identity

Let $f: \quad A \longrightarrow B$ and let $f^{-1}: \quad B \longrightarrow A$

be an **inverse** to f, then the following hold

$$(f \circ f^{-1})(a) = f^{-1}(f(a)) = I(a) = a, \quad \text{for all} \quad a \in A$$

$$(f^{-1} \circ f(b)) = f(f^{-1}(b) = I(b) = b, \quad \text{for all} \quad b \in B$$

Functions: Image and Inverse Image

Image

Given a function $f: \ X \longrightarrow \ Y$ and a set $A \subseteq X$

The set

$$f[A] = \{y \in Y : \quad \exists x \ (x \in A \ \cap \ y = f(x))\}$$

is called an **image** of the set $A \subseteq X$ **under** the function f

We write

$$y \in f[A] \quad \text{if and only if} \quad \text{there is} \quad x \in A \text{ and } y = f(x)$$

Other symbols used to denote the **image** are

$$f^{\rightarrow}(A) \quad \text{or} \quad f(A)$$

Functions: Image and Inverse Image

Inverse Image

Given a function $f : \ X \longrightarrow Y$ and a set $B \subseteq Y$

The set

$$f^{-1}[B] = \{x \in X : \ f(x) \in B\}$$

is called an **inverse image** of the set $B \subseteq Y$ **under** the function f

We write

$$x \in f^{-1}[B] \quad \text{if and only if} \quad f(x) \in B$$

Other symbol used to denote the **inverse image** are

$$f^{-1}(B) \quad \text{or} \quad f^{\leftarrow}(B)$$

Sequences

Definition

A **sequence** of elements of a set A is any **function** from the set of natural numbers N into the set A, i.e. any function

$$f: \quad N \longrightarrow A$$

Any $f(n) = a_n$ is called **n-th term** of the **sequence** f

Notations

$$f = \{a_n\}_{n \in N}, \quad \{a_n\}_{n \in N}, \quad \{a_n\}$$

Sequences Example

Example

We define a **sequence f** of real numbers R as follows

$$f: \quad N \longrightarrow R$$

such that

$$f(n) = n + \sqrt{n}$$

We also use a shorthand notation for the function f and write it as

$$a_n = n + \sqrt{n}$$

Sequences Example

We often write the function $f = \{a_n\}$ in an even shorter and **informal** form as

$$a_0 = 0, \quad a_1 = 1 + 1 = 2, \quad a_2 = 2 + \sqrt{2} \ldots\ldots\ldots$$

or even as

$$0, \quad 2, \quad 2 + \sqrt{2}, \; 3 + \sqrt{3}, \; \ldots\ldots\ldots\ldots n + \sqrt{n} \ldots\ldots\ldots$$

Observations

Observation 1

By definition, **sequence** of elements of any set is always infinite (countably infinite) because the domain of the **sequence** function f is a set N of **natural numbers**

Observation 2

We can enumerate elements of a **sequence** by any **infinite** subset of N

We usually take a set $N - \{0\}$ as a **sequence** domain (enumeration)

Observations

Observation 3

We can choose as a set of indexes of a **sequence** any countably infinite set T, i. e, **not only** the set N of natural numbers

We often choose $T = N - \{0\} = N^+$, i.e we consider **sequences** that "start" with $n = 1$

In this case we write sequences as

$$a_1, \quad a_2, \quad a_3, \quad ..... \quad a_n, \ \dots \ \dots$$

Finite Sequences

Finite Sequence

Given a finite set $K = \{1, 2, \ldots, n\}$, for $n \in N$ and any set A

Any function

$$f: \{1, 2, ...n\} \longrightarrow A$$

is called a **finite sequence** of elements of the set A

of the **length** n

Case n=0

In this case the function f is an empty set and we call it an **empty sequence**

We denote the empty sequence by e

Example

Consider a sequence given by a formula

$$a_n = \frac{n}{(n-2)(n-5)}$$

The domain of the function $f(n) = a_n$ is the set $N - \{2, 5\}$ and the **sequence** f is a function

$$f: \quad N - \{2, 5\} \ \rightarrow \ R$$

The **first** elements of the **sequence** f are

$$a_0 = f(0), \ a_1 = f(1), \ a_3 = f(3), \ a_4 = f(4) \ a_5 = f(5), \ a_6 = f(6), \ldots$$

Example

Let $T = \{-1, -2, 3, 4\}$ be a **finite** set and

$$f : \{-1, -2, 3, 4\} \rightarrow R$$

be a function given by a formula

$$f(n) = a_n = \frac{n}{(n-2)(n-5)}$$

f is a finite sequence of **length** 4 with elements

$$a_{-1} = f(-1), \quad a_{-2} = f(-2), \quad a_3 = f(3), \quad a_4 = f(4)$$

Family of sets

Any collection of sets is called a **family of sets**

We denote the family of sets by

$$\mathcal{F}$$

Sequence of sets

Any function

$$f: \ N \longrightarrow \mathcal{F}$$

is a **sequence of sets**, i..e a sequence where all its elements are sets

We use capital letters to denote sets and write the **sequence** of sets as

$$\{A_n\}_{n \in N}$$

Generalized Union

Generalized Union

Given a sequence $\{A_n\}_{n \in N}$ of sets

We define that **Generalized Union** of the sequence of sets as

$$\bigcup_{n \in N} A_n = \{x : \; \exists_{n \in N} \, x \in A_n\}$$

We write

$$x \in \bigcup_{n \in N} A_n \quad \text{if and only if} \quad \exists_{n \in N} \, x \in A_n$$

Generalized Intersection

Given a sequence $\{A_n\}_{n \in N}$ of sets

We define that **Generalized Intersection** of the sequence of sets as

$$\bigcap_{n \in N} A_n = \{x : \ \forall_{n \in N}\, x \in A_n\}$$

We write

$$x \in \bigcap_{n \in N} A_n \quad \text{if and only if} \quad \forall_{n \in N}\, x \in A_n$$

Indexed Family of Sets

Indexed Family of Sets

Given $\mathcal{F}$ be a family of sets

Let $T \neq \emptyset$ be any non empty set

Any function

$$f: \quad T \longrightarrow \mathcal{F}$$

is called an indexed family of sets with the set of indexes T

We write it

$$\{A_t\}_{t \in T}$$

Notice

Any sequence of sets is an indexed family of sets for T = N

Short Review

Some Simple Questions and Answers

Simple Short Questions

Here are some short **Yes**/ **No** questions

Answer them and write a short **justification** of your answer

Q1 $2^{\{1,2\}} \cap \{1,2\} \neq \emptyset$

Q2 $\{\{a,b\}\} \in 2^{\{a,b,\{a,b\}\}}$

Q3 $\emptyset \in 2^{\{a,b,\{a,b\}\}}$

Q4 Any function f from $A \neq \emptyset$ onto A, has property

$$f(a) \neq a \text{ for certain } a \in A$$

Simple Short Questions

Q5 Let $f: N \longrightarrow \mathcal{P}(N)$ be given by a formula:

$$f(n) = \{m \in N : \quad m < n^2\}$$

then $\emptyset \in f[\{0, 1, 2\}]$

Q6 Some relations

$$R \subseteq A \times B$$

are functions that map the set A into the set B

Answers to Short Questions

Q1 $2^{\{1,2\}} \cap \{1,2\} \neq \emptyset$

NO because

$$2^{\{1,2\}} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\} \cap \{1,2\} = \emptyset$$

Q2 $\{\{a,b\}\} \in 2^{\{a,b,\{a,b\}\}}$

YES because

have that $\{a,b\} \subseteq \{a,b,\{a,b\}\}$ and hence

$$\{\{a,b\}\} \in 2^{\{a,b,\{a,b\}\}}$$

by definition of the set of all subsets of a given set

Answers to Short Questions

Q2 $\{\{a, b\}\} \in 2^{\{a,b,\{a,b\}\}}$

YES other solution

We **list** all subsets of the set $\{a, b, \{a, b\}\}$,

i.e. all **elements** of the set

$$2^{\{a,b,\{a,b\}\}}$$

We start as follows

$$\{\emptyset, \{a\}, \{b\}, \{\{a, b\}\}, \ldots\ldots, \ldots\}$$

and observe that we can **stop** listing because we reached the set $\{\{a, b\}\}$

This proves that $\{\{a, b\}\} \in 2^{\{a,b,\{a,b\}\}}$

Answers to Short Questions

Q3 $\emptyset \in 2^{\{a,b,\{a,b\}\}}$

YES because for any set A, we have that $\emptyset \subseteq A$

Q4 Any function f from $A \neq \emptyset$ onto A has a property

$$f(a) \neq a \quad \text{for certain} \quad a \in A$$

NO

Take a function such that $f(a) = a$ for all $a \in A$

Obviously f is "onto" and and **there is no** $a \in A$

for which $f(a) \neq a$

Answers to Short Questions

Q5 Let $f : N \longrightarrow \mathcal{P}(N)$ be given by formula:

$f(n) = \{m \in N : \quad m < n^2\}$, then $\emptyset \in f[\{0, 1, 2\}]$

YES We evaluate

$f(0) = \{m \in N : \quad m < 0\} = \emptyset$

$f(1) = \{m \in N : \quad m < 1\} = \{0\}$

$f(2) = \{m \in N : \quad m < 2^2\} = \{0, 1, 2, 3\}$

and so by definition of $f[A]$ get that

$f[\{0, 1, 2\}] = \{\emptyset, \{0\}, \{0, 1, 2, 3\}\}$ and hence $\emptyset \in f[\{0, 1, 2\}]$

Q6 Some $R \subseteq A \times B$ are functions that map A into B

YES: Functions are special type of relations

Simple Short Questions

Q7 $\{(1,2),(a,1)\}$ is a binary relation on $\{1,2\}$

Q8 For any binary relation $R \subseteq A \times A$, the **inverse** relation R^{-1} **exists**

Q9 For any **binary relation** $R \subseteq A \times A$ that is a function, the **inverse function** R^{-1} exists

Simple Short Questions

Q10 Let $A = \{a, \{a\}, \emptyset\}$ and $B = \{\emptyset, \{\emptyset\}, \emptyset\}$
there is a function $f : A \xrightarrow[onto]{1-1} B$

Q11 Let $f : A \longrightarrow B$ and $g : B \xrightarrow{onto} A$,
then the **compositions** $(g \circ f)$ and $(f \circ g)$ **exist**

Q12 The function $f : N \longrightarrow \mathcal{P}(R)$ given by the formula:

$$f(n) = \{x \in R : \ x > \frac{ln(n^3 + 1)}{\sqrt{n + 6}}\}$$

is a **sequence**

Answers to Short Questions

Q7 $\{(1,2),(a,1)\}$ is a binary relation on $\{1,2\}$

NO because $(a,1) \notin \{1,2\} \times \{1,2\}$

Q8 For any binary relation $R \subseteq A \times A$, the inverse relation R^{-1} **exists**

YES By definition, the **inverse relation** to $R \subseteq A \times A$ is the set

$$R^{-1} = \{(b,a) : \quad (a,b) \in R\}$$

and it is a **well defined** relation in the set A

Answers to Short Questions

Q9 For any **binary relation** $R \subseteq A \times A$ that is a function, the **inverse function** R^{-1} exists

NO R must be also a $1-1$ and *onto* function

Q10 Let $A = \{a, \{a\}, \emptyset\}$ and $B = \{\emptyset, \{\emptyset\}, \emptyset\}$

there is a function $f : A \xrightarrow[onto]{1-1} B$

NO The set A has **3** elements and the set

$$B = \{\emptyset, \{\emptyset\}, \emptyset\} = \{\emptyset, \{\emptyset\}\}$$

has **2** elements and an onto function does not exists

Answers to Short Questions

Q11 Let $f: A \longrightarrow B$ and $g: B \longrightarrow^{onto} A$, then the **compositions** $(g \circ f)$ and $(f \circ g)$ **exist**

YES The composition $(f \circ g)$ **exists** because the functions $f: A \longrightarrow \mathbf{B}$ and $g: \mathbf{B} \longrightarrow^{onto} A$ **share** the same set $\mathbf{B}$

The composition $(g \circ f)$ **exists** because the functions $g: B \longrightarrow^{onto} \mathbf{A}$ and $f: \mathbf{A} \longrightarrow B$ **share** the same set $\mathbf{A}$

The information "onto" is irrelevant

Answers to Short Questions

Q12 The function $f: N \longrightarrow \mathcal{P}(R)$ given by the formula:

$$f(n) = \{x \in R : \quad x > \frac{ln(n^3 + 1)}{\sqrt{n + 6}}\}$$

is a **sequence**

YES It is a sequence as the **domain** of the function f is the set N of natural numbers and the formula for f(n) assigns to each natural number n a certain **subset** of R, i.e. an **element** of $\mathcal{P}(R)$

Discrete Mathematics Basics

PART 3: Special types of Binary Relations

SPECIAL RELATION: Equivalence Relation

Equivalence Relation

Equivalence relation

A binary relation $R \subseteq A \times A$ is an **equivalence** relation defined in the set A if and only if it is reflexive, symmetric and transitive

Symbols

We denote equivalence relation by symbols

$$\sim, \quad \approx \quad \text{or} \quad \equiv$$

We will use the symbol $\approx$ to denote the equivalence relation

Equivalence Relation

Equivalence class

Let $\approx \subseteq A \times A$ be an **equivalence** relation on A

The set

$$E(a) = \{b \in A : \quad a \approx b\}$$

is called an **equivalence class**

Symbol

The equivalence classes are usually **denoted** by

$$[a] = \{b \in A : \quad a \approx b\}$$

The element a is called a **representative** of the equivalence class $[a]$ defined in A

Partitions

Partition

A family of sets $\mathbf{P} \subseteq \mathcal{P}(A)$ is called a **partition** of the set A if and only if the following conditions hold

1. $\forall_{X \in \mathbf{P}} \, (X \neq \emptyset)$
   i.e. all sets in the partition are non-empty
2. $\forall_{X,Y \in \mathbf{P}} \, (X \cap Y = \emptyset)$
   i.e. all sets in the partition are disjoint
3. $\bigcup \mathbf{P} = A$
   i.e union of all sets from **P** is the set A

Equivalence and Partitions

Notation

$A/\approx$ denotes the set of **all equivalence** classes of the equivalence relation $\approx$, i.e.

$$A/\approx \; = \{[a] : a \in A\}$$

We prove the following theorem 1.3.1

Theorem 1

Let $A \neq \emptyset$

If $\approx$ is an **equivalence relation** on A,

then the set $A/\approx$ is a **partition** of A

Equivalence and Partitions

Theorem 1 (full statement)

Let $A \neq \emptyset$

If $\approx$ is an equivalence relation on A,

then the set $A/\approx$ is a **partition** of A, i.e.

1. $\forall_{[a]\in A/\approx}\ ([a] \neq \emptyset)$
   i.e. all equivalence classes are non-empty
2. $\forall_{[a]\neq[b]\in A/\approx}\ ([a] \cap [b] = \emptyset)$
   i.e. all different equivalence classes are disjoint
3. $\bigcup A/\approx = A$
   i.e the union of all equivalence classes is equal to the set A

Partition and Equivalence

We also prove a following

Theorem 2

For any **partition**

$$\mathbf{P} \subseteq \mathcal{P}(A) \quad \text{of the set} \quad A$$

one can **construct** a binary relation R on A such that R is an **equivalence** on A and its equivalence classes are exactly the sets of the **partition** **P**

Partition and Equivalence

Observe that we **can** consider, for any binary relation R on s set A the sets that "look" like equivalence classes i.e. that are defined as follows:

$$R(a) = \{b \in A;\ \ aRb\} = \{b \in A;\ \ (a, b) \in R\}$$

Fact 1

If the relation R is an **equivalence** on A, then the family $\{R(a)\}_{a \in A}$ is a **partition** of A

Fact 2

If the family $\{R(a)\}_{a \in A}$ is **not** a partition of A, then R is **not** an **equivalence** on A

Proof of Theorem 1

Theorem 1

Let $A \neq \emptyset$

If $\approx$ is an **equivalence relation** on A,

then the set $A/\approx$ is a **partition** of A

Proof

Let $A/\approx = \{[a] : a \in A\} = \mathbf{P}$

We must show that all sets in **P** are nonempty, disjoint, and together exhaust the set A

Proof of Theorem 1

1. All equivalence classes are nonempty,

This holds as $a \in [a]$ for all $a \in A$, reflexivity of equivalence relation

2. All different equivalence classes are disjoint

Consider two different equivalence classes $[a] \neq [b]$

Assume that $[a] \cap [b] \neq \emptyset$.

We have that $[a] \neq [b]$, thus there is an element c

such that $c \in [a]$ and $c \in [b]$

Hence $(a, c) \in \approx$ and $(c, b) \in \approx$

Since $\approx$ is **transitive**, we get $(a, b) \in \approx$

Proof of Theorem 1

Since $\approx$ is **symmetric**, we have that also $(a, b) \in \approx$

Now take any element $d \in [a]$;
then $(d, a) \in \approx$, and by **transitivity**, $(d, b) \in \approx$
Hence $d \in [b]$, so that $[a] \subseteq [b]$

Likewise $[b] \subseteq [a]$ and $[a] = [b]$ what contradicts the assumption that $[a] \neq [b]$

Proof of Theorem 1

3. To prove that

$$\bigcup A/\approx = \bigcup \mathbf{P} = A$$

we simply notice that each element $a \in A$ is in some set in $\mathbf{P}$

Namely we have by reflexivity that always

$$a \in [a]$$

This **ends** the proof of **Theorem 1**

Proof of the Theorem 2

Now we are going to prove that the **Theorem 1** can be reversed, namely that the following is also true

Theorem 2

For any **partition**

$$\mathbf{P} \subseteq \mathcal{P}(A)$$

of A, one can **construct** a binary relation R on A such that R is an **equivalence** and its equivalence classes are exactly the sets of the **partition** $\mathbf{P}$

Proof

We define a binary relation R as follows

$$R = \{(a, b) : \quad a, b \in X \text{ for some } X \in \mathbf{P}\}$$

Short Review

PART 3: Equivalence Relations - Short and Long Questions

Short Questions

Q1 Let $R \subseteq A \times A$ for $A \neq \emptyset$, then the set

$$[a] = \{b \in A : (a, b) \in R\}$$

is an equivalence class with a **representative** a

Q2 The set

$$\{(\emptyset, \emptyset), (\{a\}, \{a\}), (3, 3)\}$$

represents a **transitive** relation

Short Questions

Q3 There is an **equivalence** relation on the set

$$A = \{\{0\}, \{0, 1\}, 1, 2\}$$

with **3** equivalence classes

Q4 Let $A \neq \emptyset$ be such that there are exactly **25 partitions** of A

It is possible to define **20 equivalence** relations on A

Short Questions Answers

Q1 Let $R \subseteq A \times A$ then the set

$$[a] = \{b \in A : (a, b) \in R\}$$

is an **equivalence** class with a **representative** a

NO The set $[a] = \{b \in A : (a, b) \in R\}$ is an equivalence class only when the relation R is an **equivalence** relation

Q2 The set

$$\{(\emptyset, \emptyset), (\{a\}, \{a\}), (3, 3)\}$$

represents a **transitive** relation

YES Transitivity condition is vacuously **true**

Short Questions Answers

Q3 There is an equivalence relation on

$$A = \{\{0\}, \{0,1\}, 1, 2\}$$

with **3** equivalence classes

YES For example, a relation R defined by the partition

$$\mathbf{P} = \{\{\{0\}\}, \ \{\{0,1\}\}, \ \{1,2\}\}$$

and so By proof of **Theorem 2**

$$R = \{(a,b) : \ a, b \in X \text{ for some } X \in \mathbf{P}\}$$

i.e. $a = b = \{0\}$ or $a = b = \{0,1\}$ or $(a = 1$ and $b = 2)$

Short Questions Answers

Q4

Let $A \neq \emptyset$ be such that there are exactly **25** partitions of A
It is possible to define **2** equivalence relations on A

YES By **Theorem 2** one can define up to 25 (as many as partitions) of equivalence classes

Equivalence Relations

Some Long Questions

Some Long Questions

Q1 Consider a function $f: A \longrightarrow B$

Show that $R = \{(a, b) \in A \times A : \; f(a) = f(b)\}$

is an **equivalence** relation on A

Q2 Let $f: N \longrightarrow N$ be such that

$$f(n) = \begin{cases} 1 & \text{if } n \leq 6 \\ 2 & \text{if } n > 6 \end{cases}$$

Find equivalence classes of R from **Q1** for this particular function f

Long Questions Solutions

Q1 Consider a function $f: A \longrightarrow B$

Show that

$$R = \{(a, b) \in A \times A: \quad f(a) = f(b)\}$$

is an **equivalence** relation on A

Solution

1. R is **reflexive**

$(a, a) \in R$ for all $a \in A$ because $f(a) = f(a)$

Long Questions Solutions

2. R is **symmetric**

Let $(a, b) \in R$, by definition $f(a) = f(b)$ and $f(b) = f(a)$

Consequently $(b, a) \in R$

3. R is **transitive**

For any $a, b, c \in A$ we get that $f(a) = f(b)$ and $f(b) = f(c)$ implies that $f(a) = f(c)$

Long Questions Solutions

Q2 Let $f: N \longrightarrow N$ be such that

$$f(n) = \begin{cases} 1 & \text{if } n \leq 6 \\ 2 & \text{if } n > 6 \end{cases}$$

Find **equivalence classes** of

$$R = \{(a, b) \in A \times A : \quad f(a) = f(b)\}$$

for this particular f

Solution

We evaluate

$$[0] = \{n \in N : \ f(0) = f(n)\} = \{n \in N : \ f(n) = 1\}$$
$$= \{n \in N : \ n \leq 6\}$$

$$[7] = \{n \in N : \ f(7) = f(n)\} = \{n \in N : \ f(n) = 2\}$$
$$= \{n \in N : \ n > 6\}$$

There are **two** equivalence classes:

$$A_1 = \{n \in N : \ n \leq 6\}, \quad A_2 = \{n \in N : \ n > 6\}$$

Discrete Mathematics Basics

PART 3: Special types of Binary Relations

SPECIAL RELATIONS: Order Relations

Order Relations

We introduce now of another type of important binary relations: the order relations

Definition

$R \subseteq A \times A$ is an order relation on A iff R is 1.Reflexive, 2. Antisymmetric, and 3. Transitive, i.e. the following conditions are satisfied

1. $\forall_{a \in A} (a, a) \in R$
2. $\forall_{a,b \in A} ((a, b) \in R \cap (b, a) \in R \Rightarrow a = b)$
3. $\forall_{a,b,c \in A} ((a, b) \in R \cap (b, c) \in R \Rightarrow (a, c) \in R)$

Order Relations

Definition

$R \subseteq (A \times A)$ is a **total** order on A if and only if R is an **order** and any two elements of A are **comparable**, i.e. additionally the following condition is satisfied

4. $\forall_{a,b \in A} \; ((a,b) \in R \cup (b,a) \in R)$

Names

order relation is also called historically a partial order

total order is also called historically a linear order

Order Relations

Notations

order relations are usually denoted by $\leq$, or when we want to make a clear distinction from the natural order in sets of numbers we **denote** it by $\preceq$

Remember

We use $\leq$ as the **order** relation symbol, it is a **symbol** for any order relation, not a the **natural order** in sets of numbers, unless we say so

Posets

Definition

Given $A \neq \emptyset$ and an **order** relation defined on A

A tuple

$$(A, \leq)$$

is called a **poset**

Name **poset** stands historically for Partially Ordered Set

A **Diagram** of is a graphical representation of a poset and is defined as follows

Posets

A **Diagram** of a poset $(A, \leq)$ is a simplified graph constructed as follows

1. As the **order** relation $\leq$ is reflexive, i.e. $(a, a) \in R$ for all $a \in A$, we **draw** a **point** with symbol a instead of a point with symbol a and the loop

2. As the order relation $\leq$ is antisymmetric we **draw** a point$b$ **above** a point a connected, but without the arrows to indicate that $(a, b) \in R$

3. As the order relation is transitive, we connect points a, b, c with a line without arrows

Posets Special Elements

Special elements in a poset $(A, \leq)$ are: maximal, minimal, greatest (largest) and smallest (least) and are defined below.

Smallest (least) $a_0 \in A$ is a smallest (least) element in the poset $(A, \leq)$ iff $\forall_{a \in A}(a_0 \leq a)$

Greatest (largest) $a_0 \in A$ is a greatest (largest) element in the poset $(A, \leq)$ iff $\forall_{a \in A}(a \leq a_0)$

Posets Special Elements

Maximal (formal) $a_0 \in A$ is a maximal element in the poset $(A, \leq)$ iff $\neg \exists_{a \in A}(a_0 \leq a \cap a_0 \neq a)$

Maximal (informal) $a_0 \in A$ is a maximal element in the poset $(A, \leq)$ iff on a diagram of $(A, \leq)$ there is no element placed above a_0

Minimal (formal) $a_0 \in A$ is a minimal element in the poset $(A, \leq)$ iff $\neg \exists_{a \in A}(a \leq a_0 \cap a_0 \neq a)$

Minimal (informal) $a_0 \in A$ is a minimal element in the poset $(A, \leq)$ iff on the diagram of $(A, \leq)$ there is no element placed below a_0

Some Properties of Posets

Use Mathematical Induction to prove the following property of finite posets

Property 1 Every non-empty finite poset has at least one maximal element

Proof

Let $(A, \leq)$ be a finite, not empty poset (partially ordered set by $\leq$, such that A has n-elements, i.e. $|A| = n$

We carry the Mathematical Induction over $n \in N - \{0\}$

Reminder: an element $a_o \in A$ ia a maximal element in a poset $(A, \leq)$ iff the following is true.

$$\neg \exists_{a \in A} (a_0 \neq a \cap a_0 \leq a)$$

Inductive Proof

Base case: $n = 1$, so $A = \{a\}$ and a is maximal (and minimal, and smallest, and largest) in the poset $(\{a\}, \leq)$

Inductive step: Assume that any set A such that $|A| = n$ has a maximal element;

Denote by a_0 the maximal element in $(A, \leq)$

Let B be a set with $n + 1$ elements; i.e. we can write B as

$B = A \cup \{b_0\}$ for $b_0 \notin A$, for some A with n elements

Inductive Proof

By **Inductive Assumption** the poset $(A, \leq)$ has a maximal element a_0

To show that $(B, \leq)$ has a maximal element we need to consider 3 cases.

1. $b_0 \leq a_0$; in this case a_0 is also a maximal element in $(B, \leq)$

2. $a_0 \leq b_0$; in this case b_0 is a new maximal in $(B, \leq)$

3. a_0, b_0 are not compatible; in this case a_0 remains maximal in $(B, \leq)$

By Mathematical Induction we have proved that

$\forall_{n \in \in N - \{0\}} (|A| = n \Rightarrow$ A has a maximal element)

Some Properties of Posets

We just proved

Property 1 Every non-empty finite poset has at least one maximal element

Show that the **Property 1** is not true for an infinite set

Solution: Consider a poset $(Z, \leq)$, where Z is the set on integers and $\leq$ is a natural order on Z. Obviously no maximal element!

Exercise: Prove

Property 2 Every non-empty finite poset has at least one minimal element

Show that the **Property 2** is not true for an infinite set

Discrete Mathematics Basics

PART 4: Finite and Infinite Sets

Equinumerous Sets

Equinumerous sets

We call two sets A and B are equinumerous

if and only if there is a **bijection** function $f : A \longrightarrow B$,

i.e. there is f is such that

$$f : A \overset{1-1, onto}{\longrightarrow} B$$

Notation

We write $A \sim B$ to denote that the sets A and B are equinumerous and write symbolically

$$A \sim B \text{ if and only if } f : A \overset{1-1, onto}{\longrightarrow} B$$

Equinumerous Relation

Observe that for any set X, the relation $\sim$

is an **equivalence** on the set 2^X, i.e.

$$\sim \subseteq 2^X \times 2^X$$

is reflexive, symmetric and transitive and for any set A

the equivalence class

$$[A] = \{B \in 2^X : A \sim B\}$$

describes for **finite** sets all sets that have the **same number** of elements as the set A

Equinumerous Relation

Observe also that the relation $\sim$ when considered for any sets A, B **is not** an equivalence relation as its **domain** would have to be the set of all sets that **does not** exist

We extend the notion of "the same number of elements" to **any** sets by introducing the notion of cardinality of sets

Cardinality of Sets

Cardinality definition

We say that A and B have the same **cardinality** if and only if they are equipotent, i.e.

$$A \sim B$$

Cardinality notations

If sets A and B have the same **cardinality** we denote it as:

$$|A| = |B| \quad \text{or} \quad cardA = cardB$$

Cardinality of Sets

Cardinality

We put the above together in one definition

$|A| = |B|$ if and only if

there is a function f is such that

$$f:\ A \stackrel{1-1, onto}{\longrightarrow} B$$

Finite and Infinite Sets

Definition

A set A is **finite** if and only if

there is $n \in N$ and there is a function

$$f: \{0, 1, 2, ..., n-1\} \overset{1-1, onto}{\longrightarrow} A$$

In this case we have that

$$|A| = n$$

and say that the set A **has** n elements

Finite and Infinite Sets

Definition

A set A is **infinite** if and only if A is **not finite**

Here is a theorem that characterizes infinite sets

Dedekind Theorem

A set A is **infinite** if and only if

there is a **proper** subset B of the set A such that

$$|A| = |B|$$

Infinite Sets Examples

E1 Set N of natural numbers is infinite

Consider a function f given by a formula

$f(n) = 2n$ for all $n \in N$

Obviously

$$f: N \overset{1-1, onto}{\longrightarrow} 2N$$

By **Dedekind Theorem** the set N is infinite as the set 2N of even numbers are a proper subset of natural numbers N

Infinite Sets Examples

E2 Set R of real numbers is infinite

Consider a function f given by a formula

$f(x) = 2^x$ for all $x \in R$

Obviously

$$f : R \overset{1-1, onto}{\longrightarrow} R^+$$

By **Dedekind Theorem** the set R is infinite as the set R^+ of positive real numbers are a proper subset of real numbers R

Countably Infinite Sets
Cardinal Number $\aleph_0$

Definition

A set A is called **countably infinite** if and only if it has the same cardinality as the set N natural numbers, i.e. when

$$|A| = |N|$$

The **cardinality** of natural numbers N is called $\aleph_0$ (Aleph zero) and we write

$$|N| = \aleph_0$$

Countably Infinite Sets

Definition

For any set A,

$$|A| = \aleph_0 \quad \text{if and only if} \quad |A| = |N|$$

Directly from definitions we get the following

Fact 1

A set A is **countably infinite** if and only if $|A| = \aleph_0$

Countably Infinite Sets

Fact 2

A set A is **countably infinite** if and only if
all elements of A can be put in a 1-1 sequence

Other name for **countably infinite** set is
infinitely countable set and we will use both names

Countably Infinite Sets

In a case of an infinite set A and not 1-1 sequence
we can "prune" all repetitive elements to get a 1-1 sequence,
i.e. we prove the following

Fact 2a
An infinite set A is **countably infinite** if and only if
all elements of A can be put in a sequence

Countable and Uncountable Sets

Definition

A set A is **countable** if and only if A is finite or countably infinite

Fact 3

A set A is **countable** if and only if A is finite or $|A| = \aleph_0$, i.e. $|A| = |N|$

Countable and Uncountable Sets

Definition

A set A is **uncountable** if and only if A **is not** countable

Fact 4

A set A is **uncountable** if and only if A is infinite and $|A| \neq \aleph_0$, i.e. $|A| \neq |N|$

Fact 5

A set A is **uncountable** if and only if its elements **can not** be put into a sequence

Proof proof follows directly from definition and Facts 2, 4

Countably Infinite Sets

We have proved the following

Fact 2a

An infinite set A is **countably infinite** if and only if all elements of A can be put in a sequence

We use it now to prove two **theorems** about countably infinite sets

Countably Infinite Sets

Union Theorem

Union of two countably infinite sets is a countably infinite set

Proof

Let A, B be two disjoint infinitely countable sets

By Fact 2 we can list their elements as 1-1 sequences

$$A: \quad a_0,\ a_1,\ a_2, \ldots \quad \text{and} \quad B: \quad b_0,\ b_1,\ b_2, \ldots$$

and their **union** can be listed as 1-1 sequence

$$A \cup B: \quad a_0,\ b_0,\ a_1,\ b_1,\ a_2, b_2, \ldots, \ldots$$

In a case not disjoint sets we proceed the same and then "prune" all repetitive elements to get a 1-1 sequence

Countably Infinite Sets

Product Theorem

Cartesian Product of two countably infinite sets is a countably infinite set

Proof

Let A, B be two infinitely countable sets

By Fact 2 we can list their elements as 1-1 sequences

$$A : \ a_0, \ a_1, \ a_2, \ldots \quad \text{and} \quad B : \ b_0, \ b_1, \ b_2, \ldots$$

We list their **Cartesian Product** $A \times B$ as an infinite table

$(a_0, b_0), \ (a_0, b_1), \ (a_0, b_2), \ (a_0, b_3), \ \ldots$

$(a_1, b_0), \ (a_1, b_1), \ (a_1, b_2), \ (a_1, b_3), \ \ldots$

$(a_2, b_0), \ (a_2, b_1), \ (a_2, b_2), \ (a_2, b_3), \ \ldots$

$(a_3, b_0), \ (a_3, b_1), \ (a_3, b_2), \ (a_3, b_3), \ \ldots$

$\ldots, \quad \ldots, \quad \ldots, \quad \ldots, \quad \ldots, \quad \ldots,$

Cartesian Product Theorem Proof

Observe that even if the table is infinite each of its **diagonals** is **finite**

$(a_0, b_0),\ (a_0, b_1),\ (a_0, b_2),\ (a_0, b_3),\ (a_0, b_4),\ \ldots, \ldots$

$(a_1, b_0),\ (a_1, b_1),\ (a_1, b_2),\ (a_1, b_3),\ \ldots$

$(a_2, b_0),\ (a_2, b_1),\ (a_2, b_2),\ (a_2, b_3),\ \ldots$

$(a_3, b_0),\ (a_3, b_1),\ (a_3, b_2),\ (a_3, b_3),\ \ldots$

$\ldots, \quad \ldots, \quad \ldots, \quad \ldots,$

We **list** now elements of $A \times B$ one **diagonal** after the other
Each **diagonal** is finite, so now we know when one finishes and other starts

Cartesian Product Theorem Proof

$A \times B$ becomes now the following sequence

(a_0, b_0),
(a_1, b_0), (a_0, b_1),
(a_2, b_0), (a_1, b_1), (a_0, b_2),
(a_3, b_0), (a_2, b_1), (a_1, b_2), (a_0, b_3),
(a_3, b_1), (a_2, b_2), (a_1, b_3), (a_0, b_4), $\ldots$,
$\ldots$, $\ldots$, $\ldots$, $\ldots$,

We prove by Mathematical induction that the sequence is **well defined** for all $n \in N$ and hence that $|A \times B| = |N|$
It **ends** the proof of the **Product Theorem**

Union and Cartesian Product Theorems

Observe that the both **Union** and **Product Theorems** can be generalized by Mathematical Induction to the case of Union or Cartesian Products of **any finite** number of sets

Uncountable Sets

Theorem 1

The set R of real numbers is **uncountable**

Proof

We first prove (homework problem 1.5.11) the following

Lemma 1

The set of all real numbers in the interval [0,1] is **uncountable**

Then we use the Lemma 2 below (to be proved it as an exercise) and the fact that $[0, 1] \subseteq R$ and this **ends** the proof

Lemma 2 For any sets A,B such that $B \subseteq A$ and B is **uncountable** we have that also the set A is **uncountable**

Special Uncountable Sets

Cardinal Number $\mathcal{C}$ - Continuum

We denote by $\mathcal{C}$ the cardinality of the set R of real numbers

$\mathcal{C}$ is a new **cardinal number** called continuum and we write

$$|R| = \mathcal{C}$$

Definition

We say that a set A has **cardinality** $\mathcal{C}$ (continuum)

if and only if $|A| = |R|$

We write it

$$|A| = \mathcal{C}$$

Sets of Cardinality $\mathcal{C}$

Example

The set of positive real numbers R^+ has cardinality $\mathcal{C}$ because a function f given by the formula

$$f(x) = 2^x \text{ for all } x \in R$$

is 1-1 function and maps R **onto** the set R^+

Sets of Cardinality $\mathcal{C}$

Theorem 2

The set 2^N of all subsets of natural numbers is **uncountable**

Proof

We will prove it in the PART 5.

Theorem 3

The set 2^N has cardinality $\mathcal{C}$, i.e.

$$|2^N| = \mathcal{C}$$

Proof

The proof of this theorem is not trivial and is not in the scope of this course

Cantor Theorem

Cantor Theorem (1891)

For any set A,

$$|A| < |2^A|$$

where we **define**

$|A| \leq |B|$ if and only if there is a function $f: \quad A \xrightarrow{1-1} B$

$|A| < |B|$ if and only if $|A| \leq |B|$ and $|A| \neq |B|$

Cantor Theorem

Directly from the definition we have the following

Fact 6

If $A \subseteq B$ then $|A| \leq |B|$

We know that $|N| = \aleph_0$, $\mathcal{C} = |R|$, and $N \subseteq R$ hence from Fact 6, $\aleph_0 \leq \mathcal{C}$, but $\aleph_0 \neq \mathcal{C}$, as the set N is **countable** and the set R is **uncountable**

Hence we proved

Fact 7

$$\aleph_0 < \mathcal{C}$$

Uncountable Sets of Cardinality Greater then $\mathcal{C}$

By **Cantor Theorem** we have that

$$|N| < |\mathcal{P}(N)| < |\mathcal{P}(\mathcal{P}(N))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(N)))| < \dots$$

All sets

$$\mathcal{P}(\mathcal{P}(N)), \quad \mathcal{P}(\mathcal{P}(\mathcal{P}(N))) \quad \dots$$

are **uncountable** with **cardinality greater** then $\mathcal{C}$, as by Theorem 3, Fact 7, and **Cantor Theorem** we have that

$$\aleph_0 < \mathcal{C} < |\mathcal{P}(\mathcal{P}(\mathcal{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{N})))| < \dots$$

Here are some basic **Theorem** and **Facts**

Union 1

Union of two infinitely countable (of cardinality $\aleph_0$) sets is an infinitely countable set

This means that

$$\aleph_0 + \aleph_0 = \aleph_0$$

Union 2

Union of a finite (of cardinality n) set and infinitely countable (of cardinality $\aleph_0$) set is an infinitely countable set

This means that

$$\aleph_0 + n = \aleph_0$$

Union 3

Union of an infinitely countable (of cardinality $\aleph_0$) set

and a set of the same cardinality as real numbers i.e. of the cardinality C has the same cardinality as the set of real numbers

This means that

$$\aleph_0 + C = C$$

Union 4 Union of two sets of cardinality the same as real numbers (of cardinality C) has the same cardinality as the set of real numbers

This means that

$$C + C = C$$

Product 1

Cartesian Product of two infinitely countable sets is an infinitely countable set

$$\aleph_0 \cdot \aleph_0 = \aleph_0$$

Product 2

Cartesian Product of a non-empty finite set and an infinitely countable set is an infinitely countable set

$$n \cdot \aleph_0 = \aleph_0 \quad \text{for} \quad n > 0$$

Product 3

Cartesian Product of an infinitely countable set and an uncountable set of cardinality C has the cardinality C

$$\aleph_0 \cdot C = C$$

Product 4

Cartesian Product of two uncountable sets of cardinality C has the cardinality C

$$C \cdot C = C$$

Countable and Uncountable Sets

Power 1

The set 2^N of all subsets of natural numbers (or of any countably infinite set) is uncountable set of cardinality C , i.e. has the same cardinality as the set of real numbers

$$2^{\aleph_0} = C$$

Power 2

There are C of all functions that map N into N

Power 3

There are C possible **sequences** that can be form out of an infinitely countable set

$$\aleph_0^{\aleph_0} = C$$

Power 4

The set of **all functions** that map R into R has the cardinality C^C

Power 5

The set of **all real functions** of one variable has the same cardinality as the set of **all subsets** of real numbers

$$C^C = 2^C$$

Countable and Uncountable Sets

Theorem 4

$$n < \aleph_0 < C$$

Theorem 5

For any non empty, finite set A, the set A^* of all **finite sequences** formed out of A is countably infinite, i.e

$$|A^*| = \aleph_0$$

We write it as

$$\text{If} \quad |A| = n, \; n \geq 1, \quad \text{then} \; |A^*| = \aleph_0$$

Simple Short Questions

Simple Short Questions

Q1 Set A is uncountable iff $A \subseteq R$ (R is the set of real numbers)

Q2 Set A is countable iff $N \subseteq A$ where N is the set of natural numbers

Q3 The set 2^N is infinitely countable

Q4 The set $A = \{\{n\} \in 2^N : n^2 + 1 \leq 15\}$ is **infinite**

Q5 The set $A = \{(\{n\}, n) \in 2^N \times N : 1 \leq n \leq n^2\}$ is **infinitely countable**

Q6 Union of an infinite set and a finite set is an infinitely countable set

Answers to Simple Short Questions

Q1 Set A is uncountable if and only if $A \subseteq R$ (R is the set of real numbers)

NO

The set 2^R is **uncountable**, as $|R| < |2^R|$ by **Cantor Theorem**, but 2^R **is not** a subset of R

Also for example. $N \subseteq R$ and N **is not** uncountable

Answers to Simple Short Questions

Q2 Set A is **countable** if and only if $N \subseteq A$, where N is the set of natural numbers

NO

For example, the set $A = \{\emptyset\}$ is countable as it is finite, but

$$N \not\subseteq \{\emptyset\}$$

In fact, A can be any **finite** set

or any A can be any **infinite** set that does not include N, for example,

$$A = \{\{n\} : \ n \in N\}$$

Answers to Simple Short Questions

Q3 The set 2^N is infinitely countable

NO

$|2^N| = |R| = \mathcal{C}$ and hence 2^N is **uncountable**

Q4

The set $A = \{\{n\} \in 2^N : n^2 + 1 \leq 15\}$ is **infinite**

NO

The set $\{n \in N : n^2 + 1 \leq 15\} = \{0, 1, 2, 3\}$,

Hence the set $A = \{\{0\}, \{1\}, \{2\}, \{3\}\}$ is **finite**

Answers to Simple Short Questions

Q5 The set $A = \{(\{n\}, n) \in 2^N \times N : 1 \leq n \leq n^2\}$ is **infinitely countable** (countably infinite)

YES

Observe that the condition $n \leq n^2$ holds for all $n \in N$,

so the set $B = \{n : n \leq n^2\}$ is **nfinitely countable**

The set $C = \{(\{n\} \in 2^N : 1 \leq n \leq n^2\}$ is also

infinitely countable as the function given by a formula

$f(n) = \{n\}$ is $1 - 1$ and maps B onto C, i.e $|B| = |C|$

The set $A = C \times B$ is hence **infinitely countable** as the Cartesian Product of two infinitely countable sets

Discrete Mathematics Basics

PART 5: Fundamental Proof Techniques

1. Mathematical Induction
2. The Pigeonhole Principle
3. The Diagonalization Principle

Mathematical Induction Applications
Examples

Counting Functions Theorem

For any finite, non empty sets A, B, there are

$$|B|^{|A|}$$

functions that map A into B

Proof

We conduct the proof by Mathematical Induction over the **number of elements** of the set A, i.e. over $n \in N - \{0\}$, where $n = |A|$

Counting Functions Theorem Proof

Base case $n = 1$

We have hence that $|A| = 1$ and let $|B| = m, \quad m \geq 1$, i.e.

$$A = \{a\} \quad \text{and} \quad B = \{b_1, ...b_m\}, \quad m \geq 1$$

We have to prove that there are

$$|B|^{|A|} = m^1$$

functions that map A into B

The **base case** holds as there are exactly $m^1 = m$

functions $f : \{a\} \longrightarrow \{b_1, ...b_m\}$ defined as follows

$$f_1(a) = b_1, \ f_2(a) = b_2, \ ...., \ f_m(a) = b_m$$

Counting Functions Theorem Proof

Inductive Step

Let $A = A_1 \cup \{a\}$ for $a \notin A_1$ and $|A_1| = n$

By inductive assumption, there are m^n functions

$$f : A \longrightarrow B = \{b_1, ...b_m\}$$

We **group** all functions that map A_1 as follows

Group 1 contains all functions f_1 such that

$$f_1 : A \longrightarrow B$$

and they have the following property

$$f_1(a) = b_1, \quad f_1(x) = f(x) \quad \text{for all} \quad f : A \longrightarrow B \quad \text{and} \quad x \in A_1$$

By inductive assumption there are m^n functions in the **Group** 1

Inductive Step

We define now a **Group** i, for $1 \leq i \leq m$, $m = |B|$ as follows

Each **Group** i contains all functions f_i such that

$$f_i : A \longrightarrow B$$

and they have the following property

$$f_i(a) = b_1, \quad f_i(x) = f(x) \quad \text{for all} \quad f : A \longrightarrow B \quad \text{and} \quad x \in A_1$$

By inductive assumption there are m^n functions in each of the **Group** i

There are $m = |B|$ groups and each of them has m^n elements, so all together there are

$$m(m^n) = m^{n+1}$$

functions, what **ends the proof**

Mathematical Induction Applications
Pigeonhole Principle

Pigeonhole Principle Theorem

If A and B are non-empy finite sets and $|A| > |B|$,

then **there is no** one-to one function from A to B

Proof

We conduct the proof by by Mathematical Induction over

$n \in N - \{0\}$, where $n = |B|$ and $B \neq \emptyset$

Base case $n = 1$

Suppose $|B| = 1$, that is, $B = \{b\}$, and $|A| > 1$.

If $f : A \longrightarrow \{b\}$,

then there are at least two distinct elements $a_1, a_2 \in A$, such that $f(a_1) = f(a_2) = \{b\}$

Hence the function f **is not** one-to one

Pigeonhole Principle Proof

Inductive Assumption

We assume that any $f : A \longrightarrow B$ is **not one-to one** provided

$$|A| > |B| \quad \text{and} \quad |B| \leq n, \quad \text{where} \quad n \geq 1$$

Inductive Step

Suppose that $f : A \longrightarrow B$ is such that

$$|A| > |B| \quad \text{and} \quad |B| = n + 1$$

Choose some $b \in B$

Since $|B| \geq 2$ we have that $B - \{b\} \neq \emptyset$

Pigeonhole Principle Proof

Consider the set $f^{-1}(\{b\}) \subseteq A$. We have two cases

1. $|f^{-1}(\{b\})| \geq 2$

Then by definition there are $a_1, a_2 \in A$,

such that $a_1 \neq a_2$ and $f(a_1) = f(a_2) = b$ what proves that the function f **is not** one-to one

2. $|f^{-1}(\{b\})| \leq 1$

Then we consider a function

$$g: \ A - f^{-1}(\{b\}) \longrightarrow B - \{b\}$$

such that

$$g(x) = f(x) \quad \text{for all} \quad x \in A - f^{-1}(\{b\})$$

Pigeonhole Principle Proof

Observe that the inductive assumption **applies** to the function g because $|B - \{b\}| = n$ for $|B| = n + 1$ and

$$|A - f^{-1}(\{b\})| \geq |A| - 1 \text{ for } |f^{-1}(\{b\})| \leq 1$$

We know that $|A| > |B|$, so

$$|A| - 1 > |B| - 1 = n = |B - \{b\}| \text{ and } |A - f^{-1}(\{b\})| > |B - \{b\}|$$

By the inductive assumption applied to g we get that
g **is not** one -to one
Hence $g(a_1) = g(a_2)$ for some distinct $a_1, a_2 \in A - f^{-1}(\{b\})$,
but then $f(a_1) = f(a_2)$ and f **is not** one -to one either

Pigeonhole Principle Revisited

We now formulate a bit stronger version of the the pigeonhole principle and present its slightly different proof

Pigeonhole Principle Theorem

If A and B are finite sets and $|A| > |B|$,

then **there is no** one-to one function from A to B

Proof

We conduct the proof by by Mathematical Induction over

$n \in N$, where $n = |B|$

Base case $n = 0$

Assume $|B| = 0$, that is, $B = \emptyset$. Then **there is no** function $f : A \longrightarrow B$ whatsoever; let alone a one-to one function

Pigeonhole Principle Revisited Proof

Inductive Assumption

Any function $f : A \longrightarrow B$ is **not one-to one** provided

$$|A| > |B| \quad \text{and} \quad |B| \leq n, \quad n \geq 0$$

Inductive Step

Suppose that $f : A \longrightarrow B$ is such that

$$|A| > |B| \quad \text{and} \quad |B| = n + 1$$

We have to show that f is **not one-to one** under the Inductive Assumption

Pigeonhole Principle Revisited Proof

We proceed as follows

We **choose** some element $a \in A$

Since $|A| > |B|$, and $|B| = n + 1 \geq 1$ such choice is possible

Observe now that if there is another element $a' \in A$ such that $a' \neq a$ and $f(a) = f(a')$, then obviously the function f is **not one-to one** and we are done

So, **suppose now** that the chosen $a \in A$ is **the only** element mapped by f to f(a)

Pigeonhole Principle Revisited Proof

Consider then the sets $A - \{a\}$ and $B - \{f(a)\}$
and a function

$$g : A - \{a\} \longrightarrow B - \{f(a)\}$$

such that

$$g(x) = f(x) \quad \text{for all} \quad x \in A - \{a\}$$

Observe that the inductive assumption applies to g because

$$|B - \{f(a)\}| = n \quad \text{and}$$

$$|A - \{a\}| = |A| - 1 > |B| - 1 = |B - \{f(a)\}|$$

Pigeonhole Principle Revisited Proof

Hence by the inductive assumption the function

g is **not one-to one**

Therefore, there are two distinct elements elements of

$A - \{a\}$ that are mapped by g to the same element of

$B - \{f(a)\}$

The function g is, by definition, such that

$$g(x) = f(x) \quad \text{for all} \quad x \in A - \{a\}$$

so the function f is **not one-to one** either

This **ends** the proof

Pigeonhole Principle Theorem Application

The Pigeonhole Principle Theorem is a quite simple fact but is used in a large variety of proofs including many in this course We present here just one simple application which we will use in later Chapters

Path Definition

Let $A \neq \emptyset$ and $R \subseteq A \times A$ be a binary relation in the set A

A **path** in the binary relation R is a finite sequence

$a_1, \ldots, a_n$ such that $(a_i, a_{i+1}) \in R$, for $i = 1, 2, \ldots n-1$ and $n \geq 1$

The path $a_1, \ldots, a_n$ is said to be from a_1 to a_n

The **length** of the path $a_1, \ldots, a_n$ is n

The path $a_1, \ldots, a_n$ is a **cycle** if a_i are all distinct and also $(a_n, a_1) \in R$

Pigeonhole Principle Theorem Application

Path Theorem

Let R be a binary relation on a finite set A and let $a, b \in A$

If there is a **path** from a to b in R,

then there is a **path** of length at most $|A|$

Proof

Suppose that $a_1, \dots, a_n$ is the **shortest path** from $a = a_1$ to $b = a_n$, that is, the path with the smallest length, and suppose that $n > |A|$. By **Pigeonhole Principle** there is an element in A that repeats on the path, say $a_i = a_j$ for some $1 \le i < j \le n$

But then $a_1, \dots, a_i, a_{j+1}, \dots, a_n$ is a shorter path from a to b, contradicting $a_1, \dots, a_n$ being the **shortest path**

The Diagonalization Principle

Here is yet another Principle which justifies a new important proof technique

Diagonalization Principle (Georg Cantor 1845-1918)

Let R be a binary relation on a set A , i.e.

$R \subseteq A \times A$ and let D, the diagonal set for R be as follows

$$D = \{a \in A : \ (a, a) \notin R\}$$

For each $a \in A$, let

$$R_a = \{b \in A : \quad (a, b) \in R\}$$

Then D is **distinct** from each R_a

The Diagonalization Principle Applications

Here are two theorems whose proofs are the ”classic” applications of the Diagonalization Principle

Cantor Theorem 2

Let N be the set on natural numbers

The set 2^N **is uncountable**

Cantor Theorem 3

The set of real numbers in the interval $[0, 1]$ is **uncountable**

Cantor Theorem 2 Proof

Cantor Theorem 2

Let N be the set on natural numbers

The set 2^N **is uncountable**

Proof

We apply proof by contradiction method and the Diagonalization Principle

Suppose that 2^N is **countably infinite**. That is, we assume that we can put sets of 2^N in a one-to one sequence $\{R_n\}_{n \in N}$ such that

$$2^N = \{R_0,\ R_1,\ R_2,\ \ldots\}$$

We define a binary relation $R \subseteq N \times N$ as follows

$$R = \{(i,j) :\ j \in R_i\}$$

This means that for any $i, j \in N$ we have that

$$(i,j) \in R \ \text{ if and only if } \ j \in R_i$$

Cantor Theorem 2 Proof

In particular, for any $i, j \in N$ we have that

$$(i, j) \notin R \text{ if and only if } j \notin R_i$$

and the **diagonal set** D for R is

$$D = \{n \in N : \quad n \notin R_n\}$$

By definition $D \subseteq N$, i.e.

$$D \in 2^N = \{R_0, \ R_1, \ R_2, \ \ldots\}$$

and hence

$$D = R_k \quad \text{for some } k \geq 0$$

Cantor Theorem 2 Proof

We obtain **contradiction** by asking whether $k \in R_k$ for

$$D = R_k$$

We have two cases to consider: $k \in R_k$ or $k \notin R_k$

c1 Suppose that $k \in R_k$

Since $D = \{n \in N : \ n \notin R_n\}$ we have that $k \notin D$

But $D = R_k$ and we get $k \notin R_k$

Contradiction

c2 Suppose that $k \notin R_k$

Since $D = \{n \in N : \ n \notin R_n\}$ we have that $k \in D$

But $D = R_k$ and we get $k \in R_k$

Contradiction

This ends the **proof**

Cantor Theorem 3

The set of real numbers in the interval $[0,1]$ is **uncountable**

Proof

We carry the proof by the contradiction method

We assume hat the set of real numbers in the interval $[0,1]$ is **infinitely countable**

This means, by definition , that there is a function f such that

$f: N \overset{1-1, onto}{\longrightarrow} [01]$

Let f be any such function. We write $f(n) = d_n$ and denote by

$$d_0,\ d_1,\ \ldots,\ \ d_n,\ \ \ldots,$$

a sequence of of **all elements** of $[01]$ **defined** by f

We will get a **contradiction** by showing that one can always find an element $d \in [01]$ such that $d \neq d_n$ for all $n \in N$

Cantor Theorem 3 Proof

We use **binary** representation of real numbers

Hence we assume that all numbers in the interval [01] form a one to one sequence

$$d_0 = 0.a_{00}\ a_{01}\ a_{02}\ a_{03}\ a_{04}\ \ \dots\ \ \dots$$

$$d_1 = 0.a_{10}\ a_{11}\ a_{12}\ a_{13}\ a_{04}\ \ \dots\ \ \dots$$

$$d_2 = 0.a_{20}\ a_{21}\ a_{22}\ a_{23} a_0\ \ \dots\ \ \dots$$

$$d_3 = 0.a_{30}\ a_{31}\ a_{32}\ a_{33}\ a_{04} \dots\ \ \dots$$

$$\dots\ \ \dots\ \ \dots\ \ \dots\ \ \dots\ \ \dots\ \ \dots\ \ \dots$$

where all $a_{ij} \in \{0, 1\}$

Cantor Theorem 3 Proof

We use Cantor Diagonatization idea to define an element $d \in [01]$, such that $d \neq d_n$ for all $n \in N$ as follows

For each element a_{nn} of the "diagonal"

$$a_{00},\ a_{11},\ a_{22},\ \ldots\ a_{nn},\ \ldots\ ,\ \ldots$$

of the sequence $d_0,\ d_1,\ \ldots,\ d_n,\ \ldots,$ of binary representation of all elements of the interval $[01]$ we define an element $b_{nn} \neq a_{nn}$ as

$$b_{nn} = \begin{cases} 0 & \text{if } a_{nn} = 1 \\ 1 & \text{if } a_{nn} = 0 \end{cases}$$

Cantor Theorem 3 Proof

Given such defined sequence

$$b_{00},\ b_{11},\ b_{22},\ b_{33},\ b_{44},\ \ldots\ \ldots$$

We now construct a real number d as

$$d = b_{00}\ b_{11}\ b_{22}\ b_{33}\ b_{44}\ \ \ldots\ \ldots$$

Obviously $d \in [01]$ and by the Diagonatization Principle

$d \neq d_n$ for all $n \in N$

Contradiction

This ends the **proof**

Cantor Theorem 3 Proof

Here is **another proof** of the Cantor Theorem 3

It uses, after Cantor the **decimal representation** of real numbers

In this case we assume that all numbers in the interval [01] form a one to one sequence

$$d_0 = 0.a_{00}\ a_{01}\ a_{02}\ a_{03}\ a_{04}\ \ldots\ \ldots$$

$$d_1 = 0.a_{10}\ a_{11}\ a_{12}\ a_{13}\ a_{04}\ \ldots\ \ldots$$

$$d_2 = 0.a_{20}\ a_{21}\ a_{22}\ a_{23} a_0\ \ldots\ \ldots$$

$$d_3 = 0.a_{30}\ a_{31}\ a_{32}\ a_{33}\ a_{04} \ldots\ \ldots$$

$$\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots$$

where all $a_{ij} \in \{0, 1, 2 \ldots 9\}$

Cantor Theorem 3 Proof

For each element a_{nn} of the "diagonal"

$$a_{00},\ a_{11},\ a_{22},\ \dots\ a_{nn},\ \dots\ ,\ \dots$$

we define now an element (this is not the only possible definition) $b_{nn} \neq a_{nn}$ as

$$b_{nn} = \begin{cases} 2 & \text{if } a_{nn} = 1 \\ 1 & \text{if } a_{nn} \neq 1 \end{cases}$$

We construct a real number $d \in [01]$ as

$$d = b_{00}\ b_{11}\ b_{22}\ b_{33}\ b_{44}\ \dots\ \dots$$

Discrete Mathematics Basics

PART 6: Closures and Algorithms

Closures - Intuitive

Idea

Natural numbers N are **closed** under $+$, i.e. for given two natural numbers n, m we always have that $n + m \in N$

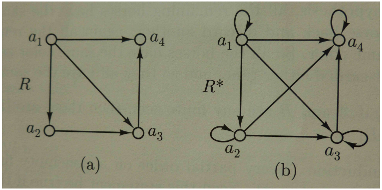
Natural numbers N are **not closed** under subtraction $-$, i.e there are two natural numbers n, m such that $n - m \notin N$, for example $1, 2 \in N$ and $1 - 2 \notin N$

Integers Z are **closed** under$-$, moreover Z is the smallest set containing N and closed under subtraction $-$

The set Z is called a **closure** of N under subtraction $-$

Closures - Intuitive

Consider the two directed graphs R (a) and R^* (b) as shown below



Observe that $R^* = R \cup \{(a_i, a_i) : \; i = 1, 2, 3, 4\} \cup \{(a_2, a_4)\}$,

$R \subseteq R^*$ and is R^* is reflexive and transitive whereas R is neither, moreover R^* is also the smallest set containing R that is reflexive and transitive

We call such relation R^* the reflexive, transitive closure of R

We define this concept formally in two ways and prove the equivalence of the two definitions

Two Definitions of R^*

Definition 1 of R^*

R^* is called a reflexive, transitive closure of R iff $R \subseteq R^*$ and is R^* is reflexive and transitive and is the smallest set with these properties

This definition is based on a notion of a **closure property** which is any property of the form " the set B is closed under relations $R_1, R_2, \ldots, R_m$"

We define it formally and prove that reflexivity and transitivity are closures properties

Hence we **justify** the name: reflexive, transitive closure of R for R^*

Two Definitions of R^*

Definition 2 of R^*

Let R be a binary relation on a set A

The **reflexive, transitive closure of** R is the relation

$$R^* = \{(a, b) \in A \times A : \quad \text{there is a path from a to b in R}\}$$

This is a much simpler definition- and algorithmically more interesting as it uses a simple notion of a path

We hence start our investigations from it- and only later introduce all notions needed for the **Definition 1** in order to prove that the R^* defined above is really what its name says: the **reflexive, transitive closure of** R

Definition 2 of R^*

We bring back the following

Path Definition

A **path** in the binary relation R is a finite sequence

$a_1, \ldots, a_n$ such that $(a_i, a_{i+1}) \in R$, for $i = 1, 2, \ldots n-1$ and $n \geq 1$

The path $a_1, \ldots, a_n$ is said to be from a_1 to a_n

The path a_1 (case when $n = 1$) always exist and is called a trivial path from a_1 to a_1

Definition 2

Let R be a binary relation on a set A

The **reflexive, transitive closure of** R is the relation

$$R^* = \{(a, b) \in A \times A : \text{ there is a path from a to b in } R\}$$

Algorithms

Definition 2 immediately suggests an following algorithm for computing the reflexive transitive closure R^* of any given binary relation R over some finite set $A = \{a_1, a_2, \ldots, a_n\}$

Algorithm 1

Initially $R^* := 0$

for $i = 1, 2, \ldots, n$ do

for each i- tuple $(b_1, \ldots, b_i) \in A^i$ do

if $b_1, \ldots, b_i$ is a **path in** R then add (b_1, b_n) to R^*

Algorithms

We also have a following much faster algorithm

Algorithm 2

Initially $R^* := R \cup \{(a_i, a_i) : \ a_i \in A\}$

for $j = 1, 2, \ldots, n$ do

for $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n$ do

if $(a_i, a_j), (a_j, a_k) \in R^*$ but $(a_i, a_k) \notin R^*$

then add (a_i, a_k) to R^*

Closure Property Formal

We introduce now formally a concept of a closure property of a given set

Definition

Let D be a set, let $n \geq 0$ and

let $R \subseteq D^{n+1}$ be a $(n+1)$-ary relation on D

Then the subset B of D is said to be **closed under** R

if $b_{n+1} \in B$ whenever $(b_1, \ldots, b_n, b_{n+1}) \in R$

Any property of the form " the set B is closed under relations $R_1, R_2, \ldots, R_m$" is called a **closure property** of B

Closure Property Examples

Observe that any function $f: D^n \longrightarrow D$ is a special relation $f \subseteq D^{n+1}$ so we have also defined what does it mean that a set $A \subseteq D$ is **closed under** the function f

E1: $+$ is a closure property of N

Adition is a function $+: N \times N \longrightarrow N$ defined by a formula $+(n, m) = n + m$, i.e. it is a **relation** $+ \subseteq N \times N \times N$ such that

$$+ = \{(n, m, n + m) : \quad n, m \in N\}$$

Obviously the set $N \subseteq N$ is (formally) closed under +
because

for any $n, m \in N$ we have that $(n, m, n + m) \in +$

Closures Property Examples

E2: $\cap$ is a closure property of 2^N

$\cap \subseteq 2^N \times 2^N \times 2^N$ is defined as

$$(A, B, C) \in \cap \quad \text{iff} \quad A \cap B = C$$

and the following is true for all $A, B, C \in 2^N$

$$\text{if } A, B \in 2^N \text{ and } (A, B, C) \in \cap \text{ then } C \in 2^N$$

Closure Property Fact1

Since relations are sets, we can speak of one relation as being closed under one or more others

We show now the following

CP Fact 1

Transitivity is a closure property

Proof

Let D be a set, let Q be a ternary relation on $D \times D$, i.e. $Q \subseteq (D \times D)^3$ be such that

$$Q = \{((a,b),(b,c),(a,c)) : \quad a,b,c \in D\}$$

Observe that for any binary relation $R \subseteq D \times D$,

R is closed under Q if and only if R is **transitive**

CP Fact1 Proof

The definition of closure of R under Q says: for any $x, y, z \in D \times D$,

$$\text{if } x, y \in R \text{ and } (x, y, z) \in Q \text{ then } z \in R$$

But $(x, y, z) \in Q$ iff $x = (a, b), y = (b, c), z = (a, c)$ and

$$(a, b), (b, c) \in R \text{ implies } (a, c) \in R$$

is a true statement for all $a, b, c \in D$ iff R is **transitive**

Closure Property Fact2

We show now the following

CP Fact 2

Reflexivity is a closure property

Proof

Let $D \neq \emptyset$, we define an **unary** relation Q' on $D \times D$, i.e. $Q' \subseteq D \times D$ as follows

$$Q' = \{(a, a) : \quad a \in D\}$$

Observe that for any R binary relation on D, i.e. $R \subseteq D \times D$ we have that

R is closed under Q' if and only if R is reflexive

Closure Property Theorem

CP Theorem

Let P be a closure property defined by relations on a set D, and let $A \subseteq D$

Then there is a **unique minimal** set B such that $B \subseteq A$ and B has property P

Two Definition of R^* Revisited

Definition 1

R^* is called a reflexive, transitive closure of R iff $R \subseteq R^*$ and is R^* is reflexive and transitive and is the smallest set with these properties

Definition 2

Let R be a binary relation on a set A

The **reflexive, transitive closure of** R is the relation

$$R^* = \{(a,b) \in A \times A : \text{ there is a path from } a \text{ to } b \text{ in } R\}$$

EquivalencyTheorem

R^* of the **Definition 2** is the same as R^* of the **Definition 1** and hence richly deserves its name reflexive, transitive closure of R

Equivalency of Two Definition of R^*

Proof Let

$$R^* = \{(a, b) \in A \times A : \quad \text{there is a path from a to b in R}\}$$

R^* is reflexive for there is a trivial path (case n=1) from a to a, for any $a \in A$

R^* is transitive as for any $a, b, c \in A$

if there is a path from a to b and a path from b to c, then there is a path from a to c

Clearly $R \subseteq R^*$ because there is a path from a to b whenever $(a, b) \in R$

Equivalency of Two Definition of R^*

Consider a set $\mathcal{S}$ of all binary relations on A that contain R and are reflexive and transitive, i.e.

$$\mathcal{S} = \{Q \subseteq A \times A : \ R \subseteq Q \text{ and } Q \text{ is reflexive and transitive } \}$$

We have just proved that $R^* \in \mathcal{S}$

We prove now that R^* is the smallest set in the poset $(\mathcal{S}, \subseteq)$, i.e. that for any $Q \in \mathcal{S}$ we have that $R^* \subseteq Q$

Equivalency of Two Definition of R^*

Assume that $(a, b) \in R^*$. By Definition 2 there is a path $a = a_1, \ldots, a_k = b$ from a to b and let $Q \in \mathcal{S}$

We prove by Mathematical Induction over the length k of the path from a to b

Base case: k=1

We have that the path is $a = a_1 = b$, i.e. $(a, a) \in R^*$ and $(a, a) \in Q$ from reflexivity of Q

Inductive Assumption:

Assume that for any $(a, b) \in R^*$ such that there is a path of length k from a to b we have that $(a, b) \in Q$

Equivalency of Two Definition of R^*

Inductive Step:

Let $(a, b) \in R^*$ be now such that there is a path of length k+1 from a to b , i.e there is a a path $a = a_1, \ldots, a_k, a_{k+1} = b$

By inductive assumption $(a = a_1, a_k) \in Q$ and by definition of the path $(a_k, a_{k+1} = b) \in R$

But $R \subseteq Q$ hence $(a_k, a_{k+1} = b) \in Q$ and $(a, b) \in Q$ by transitivity

This **ends the proof** that Definition 2 of R^* implies the Definition1

The inverse implication follows from the previously proven fact that reflexivity and transitivity are closure properties

Discrete Mathematics Basics

PART 7: Alphabets and languages

Alphabets and languages
Introduction

Data are **encoded** in the computers' memory as strings of bits or other symbols appropriate for **manipulation**

The mathematical study of the **Theory of Computation** begins with understanding of mathematics of **manipulation** of strings of symbols

We first introduce two basic notions: Alphabet and Language

Alphabet

Definition

Any finite set is called an **alphabet**

Elements of the **alphabet** are called symbols of the alphabet

This is why we also say:

Alphabet is any finite set of **symbols**

Alphabet

Alphabet Notation

We use a symbol Σ to denote the **alphabet**

Remember

Σ can be $\emptyset$ as empty set is a **finite set**

When we want to study non-empty **alphabets** we have to say so, i.e to write:

$$\Sigma \neq \emptyset$$

Alphabet Examples

E1 $\Sigma = \{\ddagger,\ \emptyset,\ \partial,\ \oint,\ \bigotimes,\ \vec{a},\ \nabla\}$

E2 $\Sigma = \{a,\ b,\ c\}$

E3 $\Sigma = \{n \in N : \quad n \leq 10^5\}$

E4 $\Sigma = \{0, 1\}$ is called a **binary alphabet**

Alphabet Examples

For simplicity and consistence we will use only as **symbols** of the alphabet letters (with indices if necessary) or other common characters when needed and specified

We also write $\sigma \in \Sigma$ for a **general** form of an element in Σ

Σ is a finite set and we will write

$$\Sigma = \{a_1, a_2, \ldots, a_n\} \quad \text{for} \quad n \geq 0$$

Finite Sequences Revisited

Definition

A **finite sequence** of elements of a set A is any function $f : \{1, 2, \ldots, n\} \longrightarrow A$ for $n \in N$

We call $f(n) = a_n$ the n-th element of the sequence f

We call n the length of the sequence

$$a_1, \; a_2, \; \ldots, \; a_n$$

Case n=0

In this case the function f is empty and we call it an **empty sequence** and denote by e

Words over Σ

Let Σ be an **alphabet**

We call finite sequences of the alphabet Σ **words** or **strings** over Σ

We denote by e the **empty word** over Σ

Some books use symbol λ for the **empty word**

Words over Σ

E5 Let $\Sigma = \{a, b\}$

We will write some words (strings) over Σ in a **shorthand** notaiton as for example

$$aaa, \; ab, \; bbb$$

instead using the formal definition:

$$f : \; \{1, 2, 3\} \; \longrightarrow \; \Sigma$$

such that $f(1) = a, f(2) = a, f(3) = a$ for the word aaa

or $g : \; \{1, 2\} \; \longrightarrow \; \Sigma$ such that $g(1) = b, g(2) = b$

for the word bb .. etc..

Words in Σ^*

Let Σ be an **alphabet**. We denote by

$$\Sigma^*$$

the set of **all finite** sequences over Σ

Elements of Σ^* are called **words** over Σ

We write $w \in \Sigma^*$ to express that w is a **word** over Σ

Symbols for words are

$$w,\ z,\ v,\ x,\ y,\ z,\ \alpha,\ \beta,\ \gamma\ \in \Sigma^*$$

$$x_1,\ x_2,\ \ldots\ \in \Sigma^* \quad y_1,\ y_2,\ \ldots\ \in \Sigma^*$$

Words in Σ^*

Observe that the set of all finite sequences include the empty sequence i.e. $e \in \Sigma^*$ and we hence have the following

Fact

For any **alphabet** Σ,

$$\Sigma^* \neq \emptyset$$

Some Short Questions and Answers

Short Questions

Q1 Let $\Sigma = \{a, b\}$

How many are there all possible **words** of length 5 over Σ ?

A1 By definition, words over Σ are finite sequences;

Hence words of a **length 5** are functions

$$f : \{1, 2, \ldots, 5\} \longrightarrow \{a, b\}$$

So we have by the **Counting Functions Theorem** that

there are 2^5 words of a length **5** over $\Sigma = \{a, b\}$

Counting Functions Theorem

Counting Functions Theorem

For any finite, non empty sets A , B, there are

$$|B|^{|A|}$$

functions that map A into B

The **proof** is in Part 5

Short Questions

Q2

Let $\Sigma = \{a_1, \ldots, a_k\}$ where $k \geq 1$

How many are there possible **words** of length $\leq n$ for $n \geq 0$ in Σ^*?

A2

By the **Counting Functions Theorem** there are

$$k^0 + k^1 + \cdots + k^n$$

words of length $\leq n$ over Σ because for each m there are k^m words of length m over $\Sigma = \{a_1, \ldots, a_k\}$ and $m = 0,\ 1 \ldots n$

Short Questions

Q3 Given an alphabet $\Sigma \neq \emptyset$

How many are there **words** in the set Σ^*?

A3

There are **infinitely countably** many **words** in Σ^* by the Theorem 5 (Lecture 2) that says: " for any non empty, finite set A, $|A^*| = \aleph_0$ "

We hence proved the following

Theorem

For any alphabet $\Sigma \neq \emptyset$, the set Σ^* of all words over Σ is **countably infinite**

Languages over Σ

Language Definition

Given an alphabet Σ, any set L such that

$$L \subseteq \Sigma^*$$

is called a **language over** Σ

Fact 1

For any alphabet Σ, any language over Σ is **countable**

Languages over Σ

Fact 2

For any alphabet $\Sigma \neq \emptyset$, there are uncountably many languages over Σ

More precisely, there are exactly $C = |R|$ of **languages** over any non - empty alphabet Σ

Languages over Σ

Fact 1

For any alphabet Σ, any language over Σ is **countable**

Proof

By definition, a set is **countable** if and only if is finite or countably infinite

1. Let $\Sigma = \emptyset$, hence $\Sigma^* = \{e\}$ and we have two languages $\emptyset$, $\{e\}$ over Σ, both finite, so **countable**
2. Let $\Sigma \neq \emptyset$, then Σ^* is countably infinite, so obviously any $L \subseteq \Sigma^*$ is finite or countably infinite, hence **countable**

Languages over Σ

Fact 2

For any alphabet $\Sigma \neq \emptyset$, there are exactly $C = |R|$ of **languages**

over any non - empty alphabet Σ

Proof

We proved that $|\Sigma^*| = \aleph_0$

By definition $L \subseteq \Sigma^*$, so there is as many languages over Σ

as all subsets of a set of cardinality $\aleph_0$ that is as many as

$2^{\aleph_0} = C$

Languages over Σ

Q4 Let $\Sigma = \{a\}$

There is $\aleph_0$ **languages over** Σ

NO

We just proved that that there is uncountably many, more precisely, exactly C languages over $\Sigma \neq \emptyset$ and we know that

$$\aleph_0 < C$$

Languages over Σ

Definition

Given an alphabet Σ and a word $w \in \Sigma^*$

We say that w has a **length** $n = |w|$ when

$$w : \{1, 2, ...n\} \longrightarrow \Sigma$$

We re-write w as

$$w : \{1, 2, |w|\} \longrightarrow \Sigma$$

Definition

Given $\sigma \in \Sigma$ and $w \in \Sigma^*$, we say $\sigma \in \Sigma$ occurs in the **j-th position** in $w \in \Sigma^*$ if and only if $w(j) = \sigma$ for $1 \leq j \leq |w|$

Some Examples

E6 Consider a word w written in a shorthand as

$$w = anita$$

By formal definition we have

$w(1) = a,\ \ w(2) = n,\ \ w(3) = i,\ \ w(4) = t,\ \ w(5) = a$

and a occurs in the 1st and 5th position

E7 Let $\Sigma = \{0, 1\}$ and $w = 01101101$ (shorthand)

Formally $w : \{1, 2, 8\} \longrightarrow \{0, 1\}$ is such that

$w(1) = 0,\ w(2) = 1,\ w(3) = 1,\ w(4) = 0,\ w(5) = 1,$

$w(6) = 1,\ w(7) = 0,\ w(8) = 1$

1 occurs in the positions 2, 3, 5, 6 and 8

0 occurs in the positions 1, 4, 7

Informal Concatenation

Informal Definition

Given an alphabet Σ and any words $x, y \in \Sigma^*$

We define informally a **concatenation** $\circ$ of words x, y as a word w obtained from x, y by writing the word x followed by the word y

We write the concatenation of words x, y as

$$w = x \circ y$$

We use the symbol $\circ$ of concatenation when it is needed formally, otherwise we will write simply

$$w = xy$$

Formal Concatenation

Definition

Given an alphabet Σ and any words $x, y \in \Sigma^*$

We define:

$$w = x \circ y$$

if and only if

1. $|w| = |x| + |y|$

2 . $w(j) = x(j)$ for $j = 1, 2, \ldots, |x|$

2 . $w(|x| + j) = j(j)$ for $j = 1, 2, \ldots, |y|$

Properties

Directly from definition we have that

$$w \circ e = e \circ w = w$$

$$(x \circ y) \circ z = x \circ (y \circ z) = x \circ y \circ z$$

Remark: we need to define a concatenation of two words and then we define

$$x_1 \circ x_2 \circ \cdots \circ x_n = (x_1 \circ x_2 \circ \cdots \circ x_{n-1}) \circ x_n$$

and prove by Mathematical Induction that

$$w = x_1 \circ x_2 \circ \cdots \circ x_n \text{ is well defined for all } n \geq 2$$

Substring

Definition

A word $v \in \Sigma^*$ is a **substring** (sub-word) of w iff there are $x, y \in \Sigma^*$ such that

$$w = xvy$$

Remark: the words $x, y \in \Sigma^*$, i.e. they can also be empty

P1 w is a substring of w

P2 e is a substring of any string (any word w)

as we have that ew = we = w

Definition Let w = xy

x is called a prefix and y is called a suffix of w

Power w^i

Definition

We define a **power** w^i of w by Mathematical Induction as follows

$$w^0 = e$$

$$w^{i+1} = w^i \circ w$$

E8

$w^0 = e, \; w^1 = w^0 \circ w = e \circ w = w, \; w^2 = w^1 \circ w = w \circ w$

E9

$anita^2 = anita^1 \circ anita = e \circ anita \circ anita = anita \circ anita$

Reversal w^R

Definition

Reversal w^R of w is defined by induction over length $|w|$ of w as follows

1. If $|w| = 0$, then $w^R = w = e$

2. If $|w| = n + 1 > 0$, then $w = ua$ for some $a \in \Sigma$, and $u \in \Sigma^*$ and we define

$$w^R = au^R \text{ for } |u| < n + 1$$

Short Definition of w^R

1. $e^R = e$

2. $(ua)^R = au^R$

Reversal Proof

We prove now as an example of Inductive proof the following simple fact

Fact

For any $w, x \in \Sigma^*$

$$(wx)^R = x^R w^R$$

Proof by Mathematical Induction over the length $|x|$ of x with $|w| = constant$

Base case n=0

$|x| = 0$, i.e. x=e and by definition

$$(we)^R = ew^R = e^R w^R$$

Reversal Proof

Inductive Assumption

$$(wx)^R = x^R w^R \quad \text{for all} \quad |x| \leq n$$

Let now $|x| = n + 1$, so $x = ua$ for certain $a \in \Sigma$ and $|u| = n$

We evaluate

$$(wx)^R = (w(ua))^R = ((wu)a)^R$$

$$=^{def} a(wu)^R =^{ind} au^R w^R =^{def} (ua)^R = x^R w^R$$

Languages over Σ

Definition

Given an alphabet Σ, any set L such that $L \subseteq \Sigma^*$ is called a **language** over Σ

Observe that $\emptyset$, Σ, Σ^* are all languages over Σ

We have proved

Theorem

Any language L over Σ, is finite or infinitely countable

Languages over Σ

Languages are **sets** so we can define them in ways we did for sets, by listing elements (for small finite sets) or by giving a **property** P(w) **defining** L , i.e. by setting

$$L = \{w \in \Sigma^* : \quad P(w)\}$$

E1

$$L_1 = \{w \in \{0,1\}^* : \quad w \quad \text{has an even number of } 0\text{'s}\}$$

E2

$$L_2 = \{w \in \{a,b\}^* : \quad w \quad \text{has } ab \text{ as a sub-string}\}$$

Languages Examples

E3

$$L_3 = \{w \in \{0,1\}^* : \quad |w| \leq 2\}$$

E4

$$L_4 = \{e,\ 0,\ 1,\ 00,\ 01,\ 11,\ 10\}$$

Observe that $\quad L_3 = L_4$

Languages Examples

Languages are **sets** so we can define set operations of union, intersection, generalized union, generalized intersection, complement, Cartesian product, ... etc ... of languages as we did for any sets

For example, given $L,\ L_1,\ L_2 \subseteq \Sigma^*$, we consider

$$L_1 \cup L_2, \quad L_1 \cap L_2, \quad L_1 - L_2,$$

$$-L = \Sigma^* - L, \quad L_1 \times L_2, , \ldots \text{ etc}$$

and we have that all properties of **algebra of sets** hold for any languages over a given alphabet Σ

Special Operations on Languages

We define now a special operation on languages, different from any of the **set** operation

Concatenation Definition

Given $L_1, \ L_2 \subseteq \Sigma^*$, a language

$$L_1 \circ L_2 = \{w \in \Sigma^* : \ \ w = xy \ \ \text{for some} \ \ x \in L_1, \ y \in L_2\}$$

is called a **concatenation** of the languages L_1 and L_2

Concatenation of Languages

The concatenation $L_1 \circ L_2$ domain issue

We can have that the languages $L_1, \ L_2$ are defined over **different domains**, i.e they have two alphabets $\Sigma_1 \neq \Sigma_2$ for

$$L_1 \subseteq {\Sigma_1}^* \quad \text{and} \quad L_2 \subseteq {\Sigma_2}^*$$

In this case we always take

$$\Sigma = \Sigma_1 \cup \Sigma_2 \quad \text{and get} \quad L_1, \ L_2 \subseteq \Sigma^*$$

Concatenation Examples

E5

Let $L_1,\ L_2$ be languages defined below

$$L_1 = \{w \in \{a,b\}^* : \quad |w| \leq 1\}$$

$$L_2 = \{w \in \{0,1\}^* : \quad |w| \leq 2\}$$

Describe the concatenation $L_1 \circ L_2$ of L_1 and L_2

Domain Σ of $L_1 \circ L_2$

We have that $\Sigma_1 = \{a,b\}$ and $\Sigma_2 = \{0,1\}$

so we take $\Sigma = \Sigma_1 \cup \Sigma_2 = \{a,b,0,1\}$ and

$$L_1 \circ L_2 \subseteq \Sigma$$

Concatenation Examples

Let L_1, L_2 be languages defined below

$$L_1 = \{w \in \{a, b\}^* : \quad |w| \leq 1\}$$

$$L_2 = \{w \in \{0, 1\}^* : \quad |w| \leq 2\}$$

We write now a **general formula** for $L_1 \circ L_2$ as follows

$$L_1 \circ L_2 = \{w \in \Sigma^* : \quad w = xy \}$$

where

$$x \in \{a, b\}^*, \quad y \in \{0, 1\}^* \text{ and } |x| \leq 1, \quad |y| \leq 2$$

Concatenation Examples

E5 revisited

Describe the concatenation of $L_1 = \{w \in \{a,b\}^* : \ |w| \leq 1\}$

and $L_2 = \{w \in \{0,1\}^* : \quad |w| \leq 2\}$

As both languages are finite, we **list** their elements and get

$L_1 = \{e,\ a,\ b\}, \quad L_2 = \{e,\ 0,\ 1,\ 01,\ 00, 11,\ 10\}$

We **describe** their concatenation as

$$L_1 \circ L_2 = \{ey : \quad y \in L_2\} \cup \{ay : \quad y \in L_2\} \cup \{by : \quad y \in L_2\}$$

Here is another **general formula** for $L_1 \circ L_2$

$$L_1 \circ L_2 = e \circ L_2 \cup (\{a\} \circ L_2) \cup (\{b\} \circ L_2)$$

Concatenation Examples

E6

Describe concatenations $L_1 \circ L_2$ and $L_2 \circ L_1$ of

$$L_1 = \{w \in \{0,1\}^* : \quad w \text{ has an even number of 0's}\}$$

and

$$L_2 = \{w \in \{0,1\}^* : \quad w = 0xx, \; x \in \Sigma^*\}$$

Here the are

$$L_1 \circ L_2 = \{w \in \Sigma^* : \; w \text{ has an odd number of 0's}\}$$

$$L_2 \circ L_1 = \{w \in \Sigma^* : \; w \text{ starts with } 0\}$$

Concatenation Examples

We have that

$L_1 \circ L_2 = \{w \in \Sigma^* : \ w$ has an odd number of 0's$\}$

$L_2 \circ L_1 = \{w \in \Sigma^* : \ w$ starts with $0\}$

Observe that

$$1000 \in L_1 \circ L_2 \quad \text{and} \quad 1000 \notin L_2 \circ L_1$$

This proves that

$$L_1 \circ L_2 \neq L_2 \circ L_1$$

We hence **proved** the following

Fact

Concatenation of languages **is not** commutative

Concatenation Examples

E8

Let $L_1, \; L_2$ be languages defined below for $\Sigma = \{0, 1\}$

$L_1 = \{w \in \Sigma^* : \quad w = x1, \;\; x \in \Sigma^*\}$

$L_2 = \{w \in \Sigma^* : \quad w = 0x, \;\; x \in \Sigma^*\}$

Describe the language $L_2 \circ L_1$

Here it is

$$L_2 \circ L_1 = \{w \in \Sigma^* : \quad w = 0xy1, \;\; x, y \in \Sigma^*\}$$

Observe that $L_2 \circ L_1$ can be also defined by a property as follows

$$L_2 \circ L_1 = \{w \in \Sigma^* : \text{w starts with } 0 \text{ and ends with} 1\}$$

Distributivity of Concatenation

Theorem

Concatenation is **distributive** over union of languages

More precisely, given languages $L, L_1, L_2, \ldots, L_n$, the following holds for any $n \geq 2$

$$(L_1 \cup L_2 \cup \cdots \cup L_n) \circ L = (L_1 \circ L) \cup \cdots \cup (L_n \circ L)$$

$$L \circ (L_1 \cup L_2 \cup \cdots \cup L_n) = (L \circ L_1) \cup \cdots \cup (L \circ L_n)$$

Proof by Mathematical Induction over $n \in N, n \geq 2$

Distributivity of Concatenation Proof

We prove the **base case** for the first equation and leave the Inductive argument and a similar proof of the second equation as an exercise

Base Case $n = 2$

We have to prove that

$$(L_1 \cup L_2) \circ L = (L_1 \circ L) \cup (L_2 \circ L)$$

$w \in (L_1 \cup L_2) \circ L$ iff (by definition of $\circ$)

$(w \in L_1 \text{ or } w \in L_2) \text{ and } w \in L$ iff (by distributivity of and over or)

$(w \in L_1 \text{ and } w \in L) \text{ or } (w \in L_2 \text{ and } w \in L)$ iff (by definition of $\circ$)

$(w \in L_1 \circ L) \text{ or } (w \in L_2 \circ L)$ iff (by definition of $\cup$)

$w \in (L_1 \circ L) \cup (L_2 \circ L)$

Kleene Star - L^*

Kleene Star L^* of a language L is yet another operation **specific** to languages

It is named after Stephen Cole Kleene (1909 -1994), an American mathematician and world famous logician who also helped lay the foundations for theoretical computer science

We define L^* as the set of all strings obtained by concatenating zero or more strings from L

Remember that concatenation of zero strings is e , and concatenation of one string is the string itself

Kleene Star - L^*

We define L^* formally as

$$L^* = \{w_1 w_2 \dots w_k : \text{for some} \quad k \geq 0 \quad \text{and} \quad w_1, \dots, w_k \in L\}$$

We also write as

$$L^* = \{w_1 w_2 \dots w_k : \quad k \geq 0, \quad w_i \in L, \quad i = 1, 2, \dots, k\}$$

or in a form of Generalized Union

$$L^* = \bigcup_{k \geq 0} \{w_1 w_2 \dots w_k : \quad w_1, \dots, w_k \in L\}$$

Remark we write xyz for $x \circ y \circ z$. We use the concatenation symbol $\circ$ when we want to stress that we talk about some properties of the concatenation

Kleene Star Properties

Here are some Kleene Star basic **properties**

P1 $e \in L^*$, for all L

Follows directly from the definition as we have case $k = 0$

P2 $L^* \neq \emptyset$, for all L

Follows directly from **P1**, as $e \in L^*$

P3 $\emptyset^* \neq \emptyset$

Because $L^* = \emptyset^* = \{e\} \neq \emptyset$

Kleene Star Properties

Some more Kleene Star basic **properties**

P4 $L^* = \Sigma^*$ for some L

Take $L = \Sigma$

P6 $L^* \neq \Sigma^*$ for some L

Take $L = \{00, 11\}$ over $\Sigma = \{0, 1\}$

We have that

$$01 \notin L^* \quad \text{and} \quad 01 \in \Sigma^*$$

Example

Observation

The property **P4** provides a quite trivial example of a language L over an alphabet Σ such that $L^* = \Sigma^*$, namely just $L = \Sigma$

A natural question arises: is there any language $L \neq \Sigma$ such that nevertheless $L^* = \Sigma^*$?

Example

Take $\Sigma = \{0,\ 1\}$ and take

$$L = \{w \in \Sigma^* : \ w \text{ has an unequal number of } 0 \text{ and } 1\}$$

Some words in and out of L are

$$100 \in L, \quad 00111 \in L \quad 100011 \notin L$$

We now **prove** that

$$L^* = \{0, 1\}^* = \Sigma^*$$

Example Proof

Given

$L = \{w \in \{0,1\}^* : \ w$ has an unequal number of 0 and $1\}$

We now **prove** that

$$L^* = \{0,1\}^* = \Sigma^*$$

Proof

By definition we have that $L \subseteq \{0,1\}^*$ and $\{0,1\}^{**} = \{0,1\}^*$

By the the following property of languages:

$$\textbf{P:} \quad \text{If } L_1 \subseteq L_2, \quad \text{then} \quad {L_1}^\star \subseteq {L_2}^\star$$

and get that

$$L^* \subseteq \{0,1\}^{**} = \{0,1\}^* \ \text{ i.e. } \ L^* \subseteq \Sigma^*$$

Example Proof

Now we have to show that $\Sigma^* \subseteq L^*$, i.e.

$$\{0,1\}^* \subseteq \{w \in 0,1^* : \ w \text{ has an unequal number of } 0 \text{ and } 1\}$$

Observe that

$0 \in L$ because 0 regarded as a string over Σ has an unequal number appearances of 0 and 1

The number of appearances of 1 is zero and the number of appearances of 0 is one

$1 \in L$ for the same reason a $0 \in L$

So we proved that $\{0,1\} \subseteq L$

We now use the property **P** and get

$$\{0, \ 1\}^* \subseteq L^* \text{ i.e } \Sigma^* \subseteq L^*$$

what **ends the proof** that $\Sigma^* = L^*$

L^* and L^+

We define

$L^+ = \{w_1 w_2 \dots w_k$: for some $k \geq 1$ and some $w_1, \dots, w_k \in L\}$

We write it also as follows

$$L^+ = \{w_1 w_2 \dots w_k : \ k \geq 1 \ \ w_i \in L, \ \ i = 1, 2, \dots, k\}$$

Properties

P1 : $L^+ = L \circ L^*$ **P2** : $e \in L^+$ iff $e \in L$

L^* and L^+

We know that

$$e \in L^* \quad \text{for all } L$$

Show that

For some language L we have that $e \in L^+$ and

for some language L we can have that $e \notin L^+$

E1

Obviously, for any L such that $e \in L$ we have that $e \in L^+$

E2

If L is such that $e \notin L$ we have that $e \notin L^+$ as L^+ does not have a case k=0

Discrete Mathematics Basics

PART 8: Finite Representation of Languages

Finite Representation of Languages
Introduction

We can represent a finite language by **finite means** for example listing all its elements

Languages are often infinite and so a natural question arises if a **finite representation** is possible and when it is possible when a language is infinite

The representation of languages by **finite specifications** is a central issue of the theory of computation

Of course we have to define first formally what do we mean by representation by finite specifications, or more precisely by a **finite representation**

Idea of Finite Representation

We start with an **example**: let

$$L = \{a\}^* \cup (\{b\} \circ \{a\}^*) = \{a\}^* \cup (\{b\}\{a\}^*)$$

Observe that by definition of Kleene's star

$$\{a\}^* = \{e,\ a,\ aa,\ aaa\ \ldots\}$$

and L is an infinite set

$$L = \{e,\ a,\ aa,\ aaa\ \ldots\} \cup \{b\}\{e,\ a,\ aa,\ aaa\ \ldots\}$$

$$L = \{e,\ a,\ aa,\ aaa\ \ldots\} \cup \{b,\ ba,\ baa,\ baaa\ \ldots\}$$

$$L = \{e,\ a,\ b, aa,\ ba,\ aaa\ baa,\ \ldots\}$$

Idea of Finite Representation

The expression $\{a\}^* \cup (\{b\}\{a\}^*)$ is built out of a finite number of **symbols**:

$$\{, \quad \}, \; (, \quad), \; *, \quad \cup$$

and describe an infinite set

$$L = \{e, \; a, \; b, aa, \; ba, \; aaa \; baa, \; \dots\}$$

We write it in a **simplified form** - we skip the set symbols $\{, \; \}$ as we know that languages are **sets** and we have

$$a^* \cup (ba^*)$$

Idea of Finite Representation

We will call such expressions as

$$a^* \cup (ba^*)$$

a **finite representation** of a language L

The idea of the finite representation is to use symbols

$$(,\ ),\ *\ ,\ \cup,\ \emptyset, \quad \text{and symbols } \sigma \in \Sigma$$

to write expressions that **describe** the language L

Example of a Finite Representation

Let L be a language defined as follows

$L = \{w \in \{0,1\}^* :$ w has **two** or **three** occurrences of 1 the **first** and the **second** of which **are not** consecutive $\}$

The language L can be expressed as

$$L = \{0\}^*\{1\}\{0\}^*\{0\} \circ \{1\}\{0\}^*(\{1\}\{0\}^* \cup \emptyset^*)$$

We will define and write the **finite representation** of L as

$$L = 0^*10^*010^*(10^* \cup \emptyset^*)$$

We call expression above (and others alike) a **regular expression**

Problem with Finite Representation

Question

Can we **finitely represent** all languages over an alphabet $\Sigma \neq \emptyset$?

Observation

O1. Different languages must have different representations

O2. **Finite representations** are finite strings over a finite set, so we have that

there are $\aleph_0$ possible **finite representations**

Problem with Finite Representation

O3. There are **uncountably** many, precisely exactly $C = |R|$) of possible languages over any alphabet $\Sigma \neq \emptyset$

Proof

For any $\Sigma \neq \emptyset$ we have proved that

$$|\Sigma^*| = \aleph_0$$

By definition of language

$$L \subseteq \Sigma^*$$

so there are as many languages as **subsets** of Σ^* that is as many as

$$|2^{\Sigma^*}| = 2^{\aleph_0} = C$$

Problem with Finite Representation

Question

Can we **finitely represent** all languages over an alphabet $\Sigma \neq \emptyset$?

Answer

No, we can't

By **O2** and **O3** there are countably many (exactly $\aleph_0$) possible **finite representations** and there are uncountably many (exactly C) possible languages over any $\Sigma \neq \emptyset$

This **proves** that

NOT ALL LANGUAGES CAN BE FINITELY REPRESENTED

Problem with Finite Representation

Moreover

There are **uncountably** many and exactly as many as Real numbers, i.e. $\mathcal{C}$ languages that **can not** be finitely represented

We can **finitely represent** only a small, countable portion of languages

We **define** and **study** here only **two** classes of languages:

REGULAR and CONTEXT FREE languages

Regular Expressions Definition

Definition

We define a $\mathcal{R}$ of **regular expressions** over an alphabet Σ as follows

$\mathcal{R} \subseteq (\Sigma \cup \{(,\ ),\ \emptyset,\ \cup,\ *\})^*$ and $\mathcal{R}$ is the smallest set such that

1. $\emptyset \in \mathcal{R}$ and $\Sigma \subseteq \mathcal{R}$, i.e. we have that

$$\emptyset \in \mathcal{R} \text{ and } \forall_{\sigma \in \Sigma} (\sigma \in \mathcal{R})$$

2. If $\alpha, \beta \in \mathcal{R}$, then

$(\alpha\beta) \in \mathcal{R}$ **concatenation**

$(\alpha \cup \beta) \in \mathcal{R}$ **union**

$\alpha^* \in \mathcal{R}$ **Kleene's Star**

Regular Expressions Theorem

Theorem

The set $\mathcal{R}$ of **regular expressions** over an alphabet Σ is countably infinite

Proof

Observe that the set $\Sigma \cup \{(,\ ),\ \emptyset,\ \cup,\ *\}$ is non-empty and **finite**, so the set $(\Sigma \cup \{(,\ ),\ \emptyset,\ \cup,\ *\})^*$ is **countably infinite**, and by definition

$$\mathcal{R} \subseteq (\Sigma \cup \{(,\ ),\ \emptyset,\ \cup,\ *\})^*$$

hence we $|\mathcal{R}| \leq \aleph_0$

The set $\mathcal{R}$ obviously includes an infinitely countable set

$$\emptyset,\ \emptyset\emptyset,\ \emptyset\emptyset\emptyset,\ \ldots,\ldots,$$

what proves that $|\mathcal{R}| = \aleph_0$

Example

Given $\Sigma = \{0, 1\}$, we have that

1. $\emptyset \in \mathcal{R}$, $1 \in \mathcal{R}$, $0 \in \mathcal{R}$

2. $(01) \in \mathcal{R}$ $1^* \in \mathcal{R}$, $0^* \in \mathcal{R}$, $\emptyset^* \in \mathcal{R}$, $(\emptyset \cup 1) \in \mathcal{R}, \ldots,$

$\ldots,$ $(((0^* \cup 1^*) \cup \emptyset)1)^* \in \mathcal{R}$

Shorthand Notation when writing regular expressions we will **keep only** essential parenthesis

For example, we will write

$((0^* \cup 1^* \cup \emptyset)1)^*$ instead of $(((0^* \cup 1^*) \cup \emptyset)1)^*$

$1^*01^* \cup (01)^*$ instead of $(((1^*0)1^*) \cup (01)^*)$

Regular Expressions and Regular Languages

We use the regular expressions from the set $\mathcal{R}$ as a **representation** of languages

Languages **represented** by regular expressions are called **regular languages**

Regular Expressions and Regular Languages

The idea of the representation is explained in the following

Example

The regular expression (written in a shorthand notion)

$$1^*01^* \cup (01)^*$$

represents a language

$$L = \{1\}^*\{0\}\{1\}^* \cup \{01\}^*$$

Definition of Representation

Definition

The **representation relation** between regular expressions and languages they **represent** is establish by a **function** $\mathcal{L}$ such that

if $\alpha \in \mathcal{R}$ is any regular expression, then $\mathcal{L}(\alpha)$ is the **language represented** by α

Definition of Representation

Formal Definition

The function $\mathcal{L} : \mathcal{R} \longrightarrow 2^{\Sigma^*}$ is defined recursively as follows

1. $\mathcal{L}(\emptyset) = \emptyset$, $\mathcal{L}(\sigma) = \{\sigma\}$ for all $\sigma \in \Sigma$

2. If $\alpha, \beta \in \mathcal{R}$, then

$$\mathcal{L}(\alpha\beta) = \mathcal{L}(\alpha) \circ \mathcal{L}(\beta)$$ **concatenation**

$$\mathcal{L}(\alpha \cup \beta) = \mathcal{L}(\alpha) \cup \mathcal{L}(\beta)$$ **union**

$$\mathcal{L}(\alpha^*) = \mathcal{L}(\alpha)^*$$ **Kleene's Star**

Regular Language Definition

Definition

A language $L \subseteq \Sigma^*$ is **regular**

if and only if

L is represented by a **regular expression**, i.e.

when there is $\alpha \in \mathcal{R}$ such that $L = \mathcal{L}(\alpha)$

where $\mathcal{L} : \mathcal{R} \longrightarrow 2^{\Sigma^*}$ is the **representation function**

We use a **shorthand notation**

$$L = \alpha \quad \text{for} \quad L = \mathcal{L}(\alpha)$$

Examples

E1

Given $\alpha \in \mathcal{R}$, for $\alpha = ((a \cup b)^* a)$

Evaluate L over an alphabet $\Sigma = \{a, b\}$, such that $L = \mathcal{L}(\alpha)$

We write

$$\alpha = ((a \cup b)^* a)$$

in the **shorthand** notation as

$$\alpha = (a \cup b)^* a$$

Examples

We evaluate $L = (a \cup b)^* a$ as follows

$$\mathcal{L}((a \cup b)^* a) = \mathcal{L}((a \cup b)^*) \circ \mathcal{L}(a) = \mathcal{L}((a \cup b)^*) \circ \{a\} =$$

$$(\mathcal{L}(a \cup b))^* \{a\} = (\mathcal{L}(a) \cup \mathcal{L}(b))^* \{a\} = (\{a\} \cup \{b\})^* \{a\}$$

Observe that

$$(\{a\} \cup \{b\})^* \{a\} = \{a,\ b\}^* \{a\} = \Sigma^* \{a\}$$

so we get

$$L = \mathcal{L}((a \cup b)^* a) = \Sigma^* \{a\}$$

$$L = \{w \in \{a,\ b\}^* :\ w \text{ ends with } a\}$$

Examples

E2 Given $\alpha \in \mathcal{R}$, for $\alpha = ((c^*a) \cup (bc^*)^*)$

Evaluate $L = \mathcal{L}(\alpha)$, i.e **describe** $L = \alpha$

We write α in the shorthand notation as

$$\alpha = c^*a \cup (bc^*)^*$$

and evaluate $L = c^*a \cup (bc^*)^*$ as follows

$$\mathcal{L}((c^*a \cup (bc^*)^*) = \mathcal{L}(c^*a) \cup (\mathcal{L}(bc^*))^* = \{c\}^*\{a\} \cup (\{b\}\{c\}^*)^*$$

and we get that

$$L = \{c\}^*\{a\} \cup (\{b\}\{c\}^*)^*$$

Examples

E3 Given $\alpha \in \mathcal{R}$, for

$$\alpha = (0^* \cup (((0^*(1 \cup (11)))((00^*)(1 \cup (11)))^*)0^*))$$

Evaluate $L = \mathcal{L}(\alpha)$, i.e **describe** the language $L = \alpha$

We write α in the **shorthand** notation as

$$\alpha = 0^* \cup 0^*(1 \cup 11)((00^*(1 \cup 11))^*)0^*$$

and evaluate

$$L = \mathcal{L}(\alpha) = 0^* \cup 0^*\{1, 11\}(00^*\{1, 11\})^*0^*$$

Observe that 00^* contains at least one 0 that separates $0^*\{1, 11\}$ on the left with $(00^*(\{1, 11\})^*$ that follows it, so we get that

$$L = \{w \in \{0, 1\}^* : w \text{ does not contain a substring } 111\}$$

Class **RL** of Regular Languages

Definition

Class **RL** of regular languages over an alphabet Σ contains all L such that $L = \mathcal{L}(\alpha)$ for certain $\alpha \in \mathcal{R}$, i.e.

$$\mathbf{RL} = \{L \subseteq \Sigma^* : \quad L = \mathcal{L}(\alpha) \quad \text{for certain} \quad \alpha \in \mathcal{R}\}$$

Theorem

There $\aleph_0$ regular languages over $\Sigma \neq \emptyset$ i.e.

$$|\mathbf{RL}| = \aleph_0$$

Proof

By definition that each regular language is $L = \mathcal{L}(\alpha)$ for certain $\alpha \in \mathcal{R}$ and the interpretation function $\mathcal{L} : \mathcal{R} \longrightarrow 2^{\Sigma^*}$ has an infinitely countable domain, hence $|\mathbf{RL}| = \aleph_0$

Class **RL** of Regular Languages

We can also think about languages in terms of **closure** and get immediately from definitions the following

Theorem

Class **RL** of regular languages is the **closure** of the set of languages

$$\{\{\sigma\} : \quad \sigma \in \Sigma\} \cup \{\emptyset\}$$

with respect to union, concatenation and Kleene Star

Languages that are NOT Regular

Given an alphabet $\Sigma \neq \emptyset$

We have just proved that there are $\aleph_0$ **regular** languages,

and we have also there are C of all languages over $\Sigma \neq \emptyset$,

so we have the following

Fact

There is C languages that are **not regular**

Natural Questions

Q1 How to prove that a given language **is regular**?

A1 Find a regular expression α, such that $L = \alpha$, i.e. $L = \mathcal{L}(\alpha)$

Languages that are NOT Regular

Q2 How to prove that a given language **is not** regular?

A2 Not easy!

There is a Theorem, called Pumping Lemma which provides a criterium for proving that a given language

is **not regular**

E1 A language

$$L = 0^*1^*$$

is **is regular** as it is given by a regular expression $\alpha = 0^*1^*$

E2 We will prove, using the Pumping Lemma that the language

$$L = \{0^n1^n : \quad n \geq 1, \quad n \in N\}$$

is **not regular**