cse581
COMPUTER SCIENCE FOUNDAMENTALS: THEORY

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Lecture 3

DISCRETE MATHEMATICS BASICS

Discrete Mathematics Basics

PART 4: Finite and Infinite Sets

PART 5: Some Fundamental Proof Techniques

Theory of Computation BASICS

PART 6: Closures and Algorithms

PART 7: Alphabets and languages

PART 8: Finite Representation of Languages

Discrete Mathematics Basics

PART 4: Finite and Infinite Sets

Equinumerous Sets

Equinumerous sets

We call two sets A and B are equinumerous

if and only if there is a **bijection** function $f: A \longrightarrow B$,

i.e. there is f is such that

$$f: A \stackrel{1-1, onto}{\longrightarrow} B$$

Notation

We write $A \sim B$ to denote that the sets A and B are equinumerous and write symbolically

$$A \sim B \quad \text{if and only if} \quad f: A \stackrel{1-1, onto}{\longrightarrow} B$$

Equinumerous Relation

Observe that for any set X, the relation $\sim$ is an **equivalence** on the set 2^X, i.e.

$$\sim \subseteq 2^X \times 2^X$$

is reflexive, symmetric and transitive and for any set A the equivalence class

$$[A] = \{B \in 2^X : A \sim B\}$$

describes for **finite** sets all sets that have the **same number** of elements as the set A

Equinumerous Relation

Observe also that the relation $\sim$ when considered for any sets A, B **is not** an equivalence relation as its **domain** would have to be the set of all sets that **does not** exist

We extend the notion of "the same number of elements" to **any** sets by introducing the notion of cardinality of sets

Cardinality of Sets

Cardinality definition

We say that A and B have the same **cardinality** if and only if they are equipotent, i.e.

$$A \sim B$$

Cardinality notations

If sets A and B have the same **cardinality** we denote it as:

$$|A| = |B| \quad \text{or} \quad cardA = cardB$$

Cardinality of Sets

Cardinality

We put the above together in one definition

$|A| = |B|$ if and only if

there is a function f is such that

$$f : A \overset{1-1, onto}{\longrightarrow} B$$

Finite and Infinite Sets

Definition

A set A is **finite** if and only if

there is $n \in N$ and there is a function

$$f : \{0, 1, 2, ..., n-1\} \overset{1-1, onto}{\longrightarrow} A$$

In this case we have that

$$|A| = n$$

and say that the set A **has** n elements

Finite and Infinite Sets

Definition

A set A is **infinite** if and only if A is **not finite**

Here is a theorem that characterizes infinite sets

Dedekind Theorem

A set A is **infinite** if and only if

there is a **proper** subset B of the set A such that

$$|A| = |B|$$

Infinite Sets Examples

E1 Set N of natural numbers is infinite

Consider a function f given by a formula

$f(n) = 2n$ for all $n \in N$

Obviously

$$f : N \overset{1-1,onto}{\longrightarrow} 2N$$

By **Dedekind Theorem** the set N is infinite as the set 2N of even numbers are a proper subset of natural numbers N

Infinite Sets Examples

E2 Set R of real numbers is infinite

Consider a function f given by a formula

$f(x) = 2^x$ for all $x \in R$

Obviously

$$f: R \xrightarrow{1-1, onto} R^+$$

By **Dedekind Theorem** the set R is infinite as the set R^+ of positive real numbers are a proper subset of real numbers R

Countably Infinite Sets
Cardinal Number $\aleph_0$

Definition

A set A is called **countably infinite** if and only if it has the same cardinality as the set N natural numbers, i.e. when

$$|A| = |N|$$

The **cardinality** of natural numbers N is called $\aleph_0$ (Aleph zero) and we write

$$|N| = \aleph_0$$

Countably Infinite Sets

Definition

For any set A,

$$|A| = \aleph_0 \quad \text{if and only if} \quad |A| = |N|$$

Directly from definitions we get the following

Fact 1

A set A is **countably infinite** if and only if $|A| = \aleph_0$

Countably Infinite Sets

Fact 2

A set A is **countably infinite** if and only if all elements of A can be put in a 1-1 sequence

Other name for **countably infinite** set is **infinitely countable** set and we will use both names

Countably Infinite Sets

In a case of an infinite set A and not 1-1 sequence
we can ”prune” all repetitive elements to get a 1-1 sequence,
i.e. we prove the following

Fact 2a
An infinite set A is **countably infinite** if and only if
all elements of A can be put in a sequence

Countable and Uncountable Sets

Definition

A set A is **countable** if and only if A is finite or countably infinite

Fact 3

A set A is **countable** if and only if A is finite or $|A| = \aleph_0$, i.e. $|A| = |N|$

Countable and Uncountable Sets

Definition

A set A is **uncountable** if and only if A **is not** countable

Fact 4

A set A is **uncountable** if and only if A is infinite and $|A| \neq \aleph_0$, i.e. $|A| \neq |N|$

Fact 5

A set A is **uncountable** if and only if its elements **can not** be put into a sequence

Proof proof follows directly from definition and Facts 2, 4

Countably Infinite Sets

We have proved the following

Fact 2a

An infinite set A is **countably infinite** if and only if all elements of A can be put in a sequence

We use it now to prove two **theorems** about countably infinite sets

Countably Infinite Sets

Union Theorem

Union of two countably infinite sets is a countably infinite set

Proof

Let A, B be two disjoint infinitely countable sets

By Fact 2 we can list their elements as 1-1 sequences

$$A: \ a_0,\ a_1,\ a_2,\ldots \quad \text{and} \quad B: \ b_0,\ b_1,\ b_2,\ldots$$

and their **union** can be listed as 1-1 sequence

$$A \cup B: \ a_0,\ b_0,\ a_1,\ b_1,\ a_2, b_2, \ldots, \ldots$$

In a case not disjoint sets we proceed the same and then ”prune” all repetitive elements to get a 1-1 sequence

Countably Infinite Sets

Product Theorem

Cartesian Product of two countably infinite sets is a countably infinite set

Proof

Let A, B be two infinitely countable sets

By Fact 2 we can list their elements as 1-1 sequences

$$A: \ a_0, \ a_1, \ a_2, \ldots \quad \text{and} \quad B: \ b_0, \ b_1, \ b_2, \ldots$$

We list their **Cartesian Product** $A \times B$ as an infinite table

$(a_0, b_0), \ (a_0, b_1), \ (a_0, b_2), \ (a_0, b_3), \ \ldots$

$(a_1, b_0), \ (a_1, b_1), \ (a_1, b_2), \ (a_1, b_3), \ \ldots$

$(a_2, b_0), \ (a_2, b_1), \ (a_2, b_2), \ (a_2, b_3), \ \ldots$

$(a_3, b_0), \ (a_3, b_1), \ (a_3, b_2), \ (a_3, b_3), \ \ldots$

$\ldots, \quad \ldots, \quad \ldots, \quad \ldots, \quad \ldots, \quad \ldots,$

Cartesian Product Theorem Proof

Observe that even if the table is infinite each of its **diagonals** is **finite**

$(a_0, b_0),\ (a_0, b_1),\ (a_0, b_2),\ (a_0, b_3),\ (a_0, b_4),\ \ldots, \ldots$

$(a_1, b_0),\ (a_1, b_1),\ (a_1, b_2),\ (a_1, b_3),\ \ldots$

$(a_2, b_0),\ (a_2, b_1),\ (a_2, b_2),\ (a_2, b_3),\ \ldots$

$(a_3, b_0),\ (a_3, b_1),\ (a_3, b_2),\ (a_3, b_3),\ \ldots$

$\ldots,\quad \ldots,\quad \ldots,\quad \ldots,$

We **list** now elements of $A \times B$ one **diagonal** after the other
Each **diagonal** is finite, so now we know when one finishes and other starts

Cartesian Product Theorem Proof

$A \times B$ becomes now the following sequence

(a_0, b_0),

(a_1, b_0), (a_0, b_1),

(a_2, b_0), (a_1, b_1), (a_0, b_2),

(a_3, b_0), (a_2, b_1), (a_1, b_2), (a_0, b_3),

(a_3, b_1), (a_2, b_2), (a_1, b_3), (a_0, b_4), $\ldots$,

$\ldots$, $\ldots$, $\ldots$, $\ldots$,

We prove by Mathematical induction that the sequence is **well defined** for all $n \in N$ and hence that $|A \times B| = |N|$

It **ends** the proof of the **Product Theorem**

Union and Cartesian Product Theorems

Observe that the both **Union** and **Product Theorems** can be generalized by Mathematical Induction to the case of Union or Cartesian Products of **any finite** number of sets

Uncountable Sets

Theorem 1

The set R of real numbers is **uncountable**

Proof

We first prove (homework problem 1.5.11) the following

Lemma 1

The set of all real numbers in the interval [0,1] is **uncountable**

Then we use the Lemma 2 below (to be proved it as an exercise) and the fact that $[0,1] \subseteq R$ and this **ends** the proof

Lemma 2 For any sets A,B such that $B \subseteq A$ and B is **uncountable** we have that also the set A is **uncountable**

Special Uncountable Sets

Cardinal Number $\mathcal{C}$ - Continuum

We denote by $\mathcal{C}$ the cardinality of the set R of real numbers

$\mathcal{C}$ is a new **cardinal number** called continuum and we write

$$|R| = \mathcal{C}$$

Definition

We say that a set A has **cardinality** $\mathcal{C}$ (continuum)

if and only if $|A| = |R|$

We write it

$$|A| = \mathcal{C}$$

Sets of Cardinality $\mathcal{C}$

Example

The set of positive real numbers R^+ has cardinality $\mathcal{C}$ because a function f given by the formula

$$f(x) = 2^x \text{ for all } x \in R$$

is 1-1 function and maps R **onto** the set R^+

Sets of Cardinality $\mathcal{C}$

Theorem 2

The set 2^N of all subsets of natural numbers is **uncountable**

Proof

We will prove it in the PART 5.

Theorem 3

The set 2^N has cardinality $\mathcal{C}$, i.e.

$$|2^N| = \mathcal{C}$$

Proof

The proof of this theorem is not trivial and is not in the scope of this course

Cantor Theorem

Cantor Theorem (1891)

For any set A,

$$|A| < |2^A|$$

where we **define**

$|A| \leq |B|$ if and only if there is a function $f: A \xrightarrow{1-1} B$

$|A| < |B|$ if and only if $|A| \leq |B|$ and $|A| \neq |B|$

Cantor Theorem

Directly from the definition we have the following

Fact 6

If $A \subseteq B$ then $|A| \leq |B|$

We know that $|N| = \aleph_0$, $\mathcal{C} = |R|$, and $N \subseteq R$ hence from Fact 6, $\aleph_0 \leq \mathcal{C}$, but $\aleph_0 \neq \mathcal{C}$, as the set N is **countable** and the set R is **uncountable**

Hence we proved

Fact 7

$$\aleph_0 < \mathcal{C}$$

Uncountable Sets of Cardinality Greater then $\mathcal{C}$

By **Cantor Theorem** we have that

$$|N| < |\mathcal{P}(N)| < |\mathcal{P}(\mathcal{P}(N))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(N)))| < \ldots$$

All sets

$$\mathcal{P}(\mathcal{P}(N)), \quad \mathcal{P}(\mathcal{P}(\mathcal{P}(N))) \quad \ldots$$

are **uncountable** with **cardinality greater** then $\mathcal{C}$, as by Theorem 3, Fact 7, and **Cantor Theorem** we have that

$$\aleph_0 < \mathcal{C} < |\mathcal{P}(\mathcal{P}(\mathcal{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{N})))| < \ldots$$

Countable and Uncountable Sets

Here are some basic **Theorem** and **Facts**

Union 1

Union of two infinitely countable (of cardinality $\aleph_0$) sets is an infinitely countable set

This means that

$$\aleph_0 + \aleph_0 = \aleph_0$$

Union 2

Union of a finite (of cardinality n) set and infinitely countable (of cardinality $\aleph_0$) set is an infinitely countable set

This means that

$$\aleph_0 + n = \aleph_0$$

Union 3

Union of an infinitely countable (of cardinality $\aleph_0$) set

and a set of the same cardinality as real numbers i.e. of the cardinality C has the same cardinality as the set of real numbers

This means that

$$\aleph_0 + C = C$$

Union 4 Union of two sets of cardinality the same as real numbers (of cardinality C) has the same cardinality as the set of real numbers

This means that

$$C + C = C$$

Product 1

Cartesian Product of two infinitely countable sets is an infinitely countable set

$$\aleph_0 \cdot \aleph_0 = \aleph_0$$

Product 2

Cartesian Product of a non-empty finite set and an infinitely countable set is an infinitely countable set

$$n \cdot \aleph_0 = \aleph_0 \quad \text{for} \quad n > 0$$

Countable and Uncountable Sets

Product 3

Cartesian Product of an infinitely countable set and an uncountable set of cardinality C has the cardinality C

$$\aleph_0 \cdot C = C$$

Product 4

Cartesian Product of two uncountable sets of cardinality C has the cardinality C

$$C \cdot C = C$$

Countable and Uncountable Sets

Power 1

The set 2^N of all subsets of natural numbers (or of any countably infinite set) is uncountable set of cardinality C , i.e. has the same cardinality as the set of real numbers

$$2^{\aleph_0} = C$$

Power 2

There are C of all functions that map N into N

Power 3

There are C possible **sequences** that can be form out of an infinitely countable set

$$\aleph_0^{\aleph_0} = C$$

Power 4

The set of **all functions** that map R into R has the cardinality C^C

Power 5

The set of **all real functions** of one variable has the same cardinality as the set of **all subsets** of real numbers

$$C^C = 2^C$$

Countable and Uncountable Sets

Theorem 4

$$n < \aleph_0 < C$$

Theorem 5

For any non empty, finite set A, the set A^* of all **finite sequences** formed out of A is countably infinite, i.e

$$|A^*| = \aleph_0$$

We write it as

$$\text{If} \quad |A| = n, \; n \geq 1, \quad \text{then} \; |A^*| = \aleph_0$$

Simple Short Questions

Simple Short Questions

Q1 Set A is uncountable iff $A \subseteq R$ (R is the set of real numbers)

Q2 Set A is countable iff $N \subseteq A$ where N is the set of natural numbers

Q3 The set 2^N is infinitely countable

Q4 The set $A = \{\{n\} \in 2^N : n^2 + 1 \leq 15\}$ is **infinite**

Q5 The set $A = \{(\{n\}, n) \in 2^N \times N : 1 \leq n \leq n^2\}$ is **infinitely countable**

Q6 Union of an infinite set and a finite set is an infinitely countable set

Answers to Simple Short Questions

Q1 Set A is uncountable if and only if $A \subseteq R$ (R is the set of real numbers)

NO

The set 2^R is **uncountable**, as $|R| < |2^R|$ by **Cantor Theorem**, but 2^R **is not** a subset of R

Also for example. $N \subseteq R$ and N **is not** uncountable

Answers to Simple Short Questions

Q2 Set A is **countable** if and only if $N \subseteq A$, where N is the set of natural numbers

NO

For example, the set $A = \{\emptyset\}$ is countable as it is finite, but

$$N \nsubseteq \{\emptyset\}$$

In fact, A can be any **finite** set

or any A can be any **infinite** set that does not include N, for example,

$$A = \{\{n\} : \quad n \in N\}$$

Answers to Simple Short Questions

Q3 The set 2^N is infinitely countable

NO

$|2^N| = |R| = \mathcal{C}$ and hence 2^N is **uncountable**

Q4

The set $A = \{\{n\} \in 2^N : n^2 + 1 \leq 15\}$ is **infinite**

NO

The set $\{n \in N : \ n^2 + 1 \leq 15\} = \{0, 1, 2, 3\}$,

Hence the set $A = \{\{0\}, \{1\}, \{2\}, \{3\}\}$ is **finite**

Answers to Simple Short Questions

Q5 The set $A = \{(\{n\}, n) \in 2^N \times N : 1 \leq n \leq n^2\}$ is **infinitely countable** (countably infinite)

YES

Observe that the condition $n \leq n^2$ holds for all $n \in N$,

so the set $B = \{n : n \leq n^2\}$ is **nfinitely countable**

The set $C = \{(\{n\} \in 2^N : 1 \leq n \leq n^2\}$ is also

infinitely countable as the function given by a formula

$f(n) = \{n\}$ is $1 - 1$ and maps B onto C, i.e $|B| = |C|$

The set $A = C \times B$ is hence **infinitely countable** as the Cartesian Product of two infinitely countable sets