cse581
Computer Science Fundamentals: Theory

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CM - Lecture 2

Chapter 1: Generalized Josephus

Binary and Relaxed Binary Solutions

From Recursive Formula to Closed Form Formula

Often the problem with a recurrent solution is in its computational complexity;

Observe that for any recursive formula R_n, in order to calculate its value for a certain n one needs to calculate (recursively) all values for R_k, $k = 1, \ldots, n-1$.

It's easy to see that for large n, this can be quite complex.

So **we would like to find** (if possible) a **non- recursive function** with a formula **f = f(n)**,

Such formula is called a **Closed Form Formula**

Provided that the **Closed Form Formula** computes **the same function** as our original recursive one.

From Recursive Formula to Closed Form Formula

We **examine classes** of Recursive Formula functions for which the it is **possible to find** corresponding **equivalent** Closed Form Formula function

Of course we have always to **prove** that Recursive Formula functions and Closed Form Formula functions we have found are **equal,** i.e. their corresponding **formulas are equivalent**.

Original Josephus Recurrence Formula

The Recurrence Formula RF J(n) is:

$J(1) = 1$

$J(2n) = 2J(n) - 1$

$J(2n + 1) = 2J(n) + 1$

where J(k) is a position of the **survivor** in Josepus Problem.

Generalized Josephus GF

We **generalized** the function J to function $f : N - \{0\} \longrightarrow N$ defined as follows

$$f(1) = \alpha$$

$$f(2n) = 2f(n) + \beta, \qquad n \geq 1$$

$$f(2n+1) = 2f(n) + \gamma, \qquad n \geq 1$$

Observe that J = f for $\alpha = 1,\ \beta = -1,\ \gamma = 1$

We call the function f a Generalized Josephus **GJ**

Original Josepus Binary Solution

We proved that the **original** J-recurrence:

J(1) = 1, J(2n) = 2J(n) - 1, J(2n+1) = 2J(n) + 1 for $n > 1$

has a beautiful **binary** CF **solution**

$$J((b_m, b_{m-1}, ...b_1, b_0)_2) = (b_{m-1}, b_{m-2}, ...b_0, b_m)_2,$$

move b_m !

where $b_m = 1$, as $n = 2^m + l$

Question: Does the Generalized Josephus GJ admits a similar solution?

Answer: YES.

Generalized Josephus **GJ**

We **generalized** the function J to function $f: N - \{0\} \longrightarrow N$ defined as follows

$$f(1) = \alpha$$

$$f(2n) = 2f(n) + \beta, \quad n \geq 1$$

$$f(2n+1) = 2f(n) + \gamma, \quad n \geq 1$$

Observe that Josephus function J = f for
$\alpha = 1, \beta = -1, \gamma = 1$

We call the function f a Generalized Josephus **GJ**

GJ in Binary Representation

We write the Generalized Josephus **GJ** function as follows

$$
\begin{aligned}
&f(1) = \alpha; \\
&f(2n + j) = 2f(n) + \beta_j \\
&\text{for } \ j = 0, 1, \qquad\qquad n \geq 1
\end{aligned}
$$

We use **Binary Representation** to find CF for **GJ**.

In order to do so, for a given $k \in N$ and its binary representation $k = (b_m, b_{m-1}, ...b_1, b_0)_2$, we first examine

$$f(k) = f((b_m, b_{m-1}, ...b_1, b_0)_2)$$

as follows.

Binary Representation for k=2n

Consider RF: $f(2n+j) = 2f(n) + \beta_j$ for $j = 0$, i.e

$$k = 2n + 0, \quad j = 0$$

The binary representation of $k = 2n$ is given as:

$$2n = (b_m, b_{m-1}, ...b_1, b_0)_2$$

$$2n = 2^m b_m + 2^{m-1} b_{m-1} + ... + 2b_1 + b_0$$

Binary Representation for k=2n

We get $b_m = 1$ and $b_0 = 0$

Hence,

$$n = 2^{m-1} b_m + ... + b_1$$

$$\mathbf{n} = (\mathbf{b_m}, \mathbf{b_{m-1}}, ...\mathbf{b_1})_\mathbf{2}$$

Question: What happens when RF: $f(2n+j) = 2f(n) + \beta_j$ for $j = 1$, i.e the the case $k = 2n+1$

Binary Representation for k=2n+1

The binary representation of k=2n +1 is given as:

$$2n + 1 = (b_m, b_{m-1}, ...b_1, b_0)_2$$

$$2n + 1 = 2^m b_m + 2^{m-1} b_{m-1} + ... + 2b_1 + b_0$$

$$b_0 = 1, b_m = 1$$

Binary Representation for k=2n+1

We get

$$2n + 1 = 2^m b_m + 2^{m-1} b_{m-1} + ... + 2b_1 + 1$$

$$2n = 2^m b_m + 2^{m-1} b_{m-1} + ... + 2b_1$$

$$n = 2^{m-1} b_m + 2^{m-1} b_{m-1} + ... + b_1$$

$$\mathbf{n = (b_m, b_{m-1}, ...b_1)_2}$$

Binary Representation

We have **proved** that whether we have a binary representation of $2n = (b_m, b_{m-1}, ...b_1, b_0)_2$
or a binary representation of 2n+1 = $(b_m, b_{m-1}, ...b_1, b_0)_2$,
the corresponding representations of n are the same:

$$\mathbf{n} = (\mathbf{b_m}, \mathbf{b_{m-1}}, ...\mathbf{b_1})_\mathbf{2}$$

Binary Fact

Binary Fact

When dealing with **binary representation** for arguments of

RF: $f(1) = \alpha, \quad f(2n) = 2f(n) + \beta_0, \quad f(2n+1) = 2f(n) + \beta_1$

we do not need to consider cases of **n** $\in$ **odd** and **n** $\in$ **even**

and we rewrite RF as

$f((b_m, b_{m-1}, ...b_1, b_0)_2) = 2f((b_m, b_{m-1}, ...b_1)_2) + \beta_{b_i}$

where

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} \qquad j = 0...m-1$$

CF in Binary Representation

Given the **GJ** recursive formula

RF: $f(1) = \alpha, \quad f(2n) = 2f(n) + \beta_0, \quad f(2n+1) = 2f(n) + \beta_1$

In order to find its CF in Binary Representation we write, by **Binary Fact**, the RF using n in **binary representation** as

$f((b_m, b_{m-1}, ...b_1, b_0)_2) = 2f((b_m, b_{m-1}, ...b_1)_2) + \beta_{b_i}$

where

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} \qquad j = 0...m-1$$

and we evaluate it recursively as follows.

CF in Binary Representation

$$\begin{aligned}
\mathbf{f((b_m, b_{m-1}, ...b_1, b_0)_2)} &= 2f((b_m, b_{m-1}, ..., b_1)_2) + \beta_{b_0} \\
&= 2(2f((b_m, b_{m-1}, ..., b_2)_2) + \beta_{b_1}) + \beta_{b_0} \\
&= 4f((b_m, b_{m-1}, ..., b_2)_2) + 2\beta_{b_1} + \beta_{b_0} \\
&\vdots \\
&= 2^m f((b_m)_2) + 2^{m-1}\beta_{b_{m-1}} + ... + 2\beta_{b_1} + \beta_{b_0} \\
&= \mathbf{2^m f(1) + 2^{m-1}\beta_{b_{m-1}} + ... + 2\beta_{b_1} + \beta_{b_0}}
\end{aligned}$$

CF in Binary Representation

We know that $f(1) = \alpha$

So we get (almost) CF formula

$$\mathbf{f((b_m, b_{m-1}, ...b_1, b_0)_2) = 2^m}\alpha + \mathbf{2^{m-1}}\beta_{\mathbf{b_{m-1}}} + ... + \mathbf{2}\beta_{\mathbf{b_1}} + \beta_{\mathbf{b_0}}$$

where

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} \qquad j = 0...m-1$$

Relaxed Binary CF

We **define** a **relaxed binary** representation as follows

$$\mathbf{2^m}\alpha + \mathbf{2^{m-1}}\beta_{\mathbf{b_{m-1}}} + ... + \beta_{\mathbf{b_0}} = (\alpha, \beta_{\mathbf{b_{m-1}}}, ...\beta_{\mathbf{b_0}})_\mathbf{2}$$

where now β_{b_k} are now any numbers, not only 0,1

We write the **relaxed binary** CF as

$$\begin{aligned} \mathbf{f((b_m, b_{m-1}, ...b_1, b_0)_2)} &= (\alpha, \beta_{\mathbf{b_{m-1}}}, ...\beta_{\mathbf{b_0}})_\mathbf{2} \\ \text{"}\textit{normal}\text{"} &= \textit{relaxed} \end{aligned}$$

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} \qquad j = 0, ..., m-1$$

Example: Original Josephus

The GJ function f becomes the **original Josephus** when $\beta_0 = -1, \beta_1 = 1$

Example

Let n = 100

Use the **relaxed binary** CF to show that f(100) = 73 =J(n) as we have already evaluated

$$n = (1\,1\,0\,0\,1\,0\,0)_2$$
$$(b_6 b_5 b_4 b_3 b_2 b_1 b_0)$$

Relaxed coordinates are

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases}$$ and hence

$$\beta_{b_j} = \begin{cases} -1 & b_j = 0 \\ 1 & b_j = 1 \end{cases}$$

Example

We have

$$\begin{aligned} n &= (\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ )_2 \\ & \quad (b_6 b_5 b_4 b_3 b_2 b_1 b_0) \end{aligned}$$

$$\begin{aligned} \mathbf{f((b_m, b_{m-1}, ...b_1, b_0)_2)} &= (\alpha, \beta_{\mathbf{b_{m-1}}}, ...\beta_{\mathbf{b_0}})_{\mathbf{2}} \\ \text{"}\textit{normal}\text{"} &= \textit{relaxed} \end{aligned}$$

$$\beta_{b_j} = \begin{cases} -1 & b_j = 0 \\ 1 & b_j = 1 \end{cases}$$

We evaluate

$$f(n) = f((\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ )_2)) =^{relax} (\alpha,\ \beta_{b_5},\ \ldots\ \beta_{b_0})$$

$$= (1, 1, -1, -1, 1, -1, -1)_2 = 64 + 32 - 16 - 8 + 4 - 2 - 1 = 73$$

Cyclic - Shift Property

We **proved** that the original J-recurrence:

J(1) = 1, J(2n) = 2J(n) - 1, J(2n+1) = 2J(n) + 1 for $n > 1$

has a beautiful binary CF solution, called **cyclic - shift property**, namely

$$J((b_m, b_{m-1}, ...b_1, b_0)_2) = (b_{m-1}, b_{m-2}, ...b_0, b_m)_2$$

We prove now that the **cyclic - shift property** holds also for the GF function f in the case when $\beta_0 = -1, \beta_1 = 1$, i.e.

$$f((b_m, b_{m-1}, ...b_1, b_0)_2) = (b_{m-1}, b_{m-2}, ...b_0, b_m)_2$$

We know that $b_m = 1$, so we have to prove that:

$$f(1, b_{m-1}, ...b_1, b_0)_2) = (b_{m-1}, b_{m-2}, ...b_0, 1)_2,$$

for f such that $\beta_0 = -1, \beta_1 = 1$

Cyclic - Shift Property for GJ

We have proved the **relaxed binary** CF solution for GJ:

$$CF: \quad f((1, b_{m-1}, ...b_1, b_0)_2) = (1, \beta_{b_{m-1}}, ...\beta_{b_0})_2$$

where $f(n)$ contains now 1 and -1 as defined by

$$\beta_{b_j} = \begin{cases} -1 & b_j = 0 \\ 1 & b_j = 1 \end{cases}$$

Example

EXAMPLE

Consider $n = (1, 0, 0, 1, 0, 0, 1)_2$

By CF we have that

$$f((1, 0, 0, 1, 0, 0, 1)_2) = (1, -1, -1, 1, -1, -1, 1)_2$$

General Observation

f transforms a BLOCK of 0's in **normal binary representation** into a BLOCK of -1's in the **relaxed representation**

$$\mathbf{f((1, 0, 0 ... 0)_2) = (1, -1, -1 ... -1)_2}$$

ONE BLOCK Transformation

We **prove** now the following relationship between relaxed and **normal** representation

ONE BLOCK **transformation**

$$(\mathbf{1}, \mathbf{-1}, \mathbf{-1} \ldots, \mathbf{-1})_{\mathbf{2}} = (\mathbf{0}, \mathbf{0}, \mathbf{0} \ldots, \mathbf{0}, \mathbf{1})_{\mathbf{2}}$$

Proof: Let $n = ((-1, -1 \ldots, -1)_2$

$$\begin{aligned}
\mathbf{n} = (\mathbf{1}, \mathbf{-1}, \mathbf{-1} \ldots, \mathbf{-1})_{\mathbf{2}} \quad &=^{def} \quad 2^m - 2^{m-1} - 2^{m-2} - \ldots - 2^1 - 2^0 \\
&= \quad 2^{m-1} - 2^{m-2} - \ldots - 2^1 - 2^0 \\
&= \quad 2^{m-2} - 2^{m-3} - \ldots - 2^1 - 2^0 \\
& \quad \vdots \\
&= \quad 2^1 - 2^0 \\
&= \quad 2 - 1 \\
&= \quad \mathbf{1} = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1})_{\mathbf{2}}
\end{aligned}$$

Many Blocks Transformation

Example for TWO BLOCKS **transformation** plus **binary shift**

$$
\begin{aligned}
f((1,0,0,1,1,0,0,1)_2) &= (1,-1,-1,1,1,-1,-1,1)_2 \\
&=^{1bt} (0,0,1,1,1,-1,-1,1)_2 \\
&=^{1bt} (0,0,1,1,0,0,1,1)_2 \\
&= (0,0,1,1,0,0,1,1)_2
\end{aligned}
$$

We know that $f((b_m, ..., b_1, b_0)_2) = (\alpha, \beta_{b_{m-1}}, ..., \beta_{b_0})_2$

OBSERVE that each block of binary digits $(1, 0..0)_2$ is **transformed** by f into $(1, -1, ...)_2$ and multiple applications of **one block transformation** transforms them **back** to $(1, 0..0)_2$, so

$$((\alpha, \beta_{\mathbf{b_{m-1}}}, \ldots, \beta_{\mathbf{b_0}})_2 =^{\mathbf{mbt}} (\mathbf{b_{m-1}}, ...\mathbf{b_1}, \mathbf{b_0}, \mathbf{1})_2$$

where mbt denotes multiple BLOK transformations, and we know that $\alpha = 1$

Cyclic - Shift Property

We now evaluate:

$$
\begin{aligned}
f((1, b_{m-1}, ..., b_1, b_0)_2) &= (\alpha, \beta_{b_{m-1}}, ..., \beta_{b_0})_2 \\
&=^{mbt} (b_{m-1}, ..., b_1, b_0, 1)_2
\end{aligned}
$$

This ends the proof of the Cyclic - Shift Property for Generalized Josephus f with $\alpha = 1,\ \beta_0 = -1,\ \beta_1 = 1$

Exercise 1

Given

$$\begin{aligned} f(1) &= 5 \\ f(2n) &= 2f(n) - 10 \\ f(2n+1) &= 2f(n) + 83 \end{aligned}$$

Exercise 1

Evaluate f(100)

Solution: just apply proper formulas!

$$\begin{aligned} \mathbf{f((b_m, b_{m-1}, ...b_1, b_0)_2)} &= (\alpha, \beta_{\mathbf{b_{m-1}}}, ...\beta_{\mathbf{b_0}})_{\mathbf{2}} \\ \text{"}\textit{normal}\text{"} &= \textit{relaxed} \end{aligned}$$

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} \qquad j = 0, ..., m-1$$

Exercise 2

Given

$$\begin{aligned} f(1) &= 5 \\ f(2n) &= 3f(n) - 10 \\ f(2n+1) &= 3f(n) + 83 \end{aligned}$$

Exercise 2

Evaluate f(100)

Observe that now we don't have proper formulas! The **GJ** solution works only for the case $2f(n)$

Goal: Generalize GJ to cover this case and develop **new closed formula CF**

Relaxed Binary Representation for GJ

We **proved** while solving the Generalized Josephus **GJ** that

RF: $f(1) = \alpha, \quad f(2n+j) = 2f(n) + \beta_j$

where $j = 0, 1$ and $n \geq 0$

has a **relaxed binary** CF formula

$$CF: \quad \mathbf{f((1, b_{m-1}, ... b_1, b_0)_2)} = (\alpha, \beta_{\mathbf{b_{m-1}}}, ... \beta_{\mathbf{b_0}})_{\mathbf{2}}$$

where β_{b_j} are defined by

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} \qquad j = 0, ..., m-1$$

GJ Generalization k-GJ and k- Representation

We **generalize** GJ to k-GJ as follows

RF: $f(1) = \alpha, \quad f(2n + j) = kf(n) + \beta_j,$

where $j = 0, 1$ and $k \geq 2$

In order to find closed formula for k-GJ we use a following generalization of the **binary** representation

$$n = (b_m, b_{m-1}, ...b_1, b_0)_k = k^m b_m + k^{m-1} b_{m-1} + ... + kb_1 + b_0$$

where $k \geq 2$, $b_m \neq 0$, each $b_i < k$

We call it a **k- representation** of n (representation of n to the base k)

Closed Formula CF for k-GJ

We are going to show that RF has a **k - representation** closed formula

$CF: \quad \mathbf{f((1, b_{m-1}, ...b_1, b_0)_2)} = (\alpha, \beta_{\mathbf{b_{m-1}}}, ...\beta_{\mathbf{b_0}})_{\mathbf{k}}$

for β_{b_j} are defined as in GJ by

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} \quad ; \quad j = 0, ..., m-1$$

and where $(\alpha, \beta_{\mathbf{b_{m-1}}}, ..., \beta_{\mathbf{b_0}})_{\mathbf{k}} = \alpha \mathbf{k^m} + \mathbf{k^{m-1}} \beta_{\mathbf{m-1}} + ... + \beta_{\mathbf{b_0}}$

is called a **relaxed k- representation**

k- Representation Closed Formula

Observe that **Binary Fact** of General Josephus GJ holds in our case and we can **rewrite**

RF: $f(1) = \alpha, \quad f(2n + j) = k f(n) + \beta_j,$

using n in **binary representation** as

RF $f((b_m, b_{m-1}, ...b_1, b_0)_2) = k f((b_m, b_{m-1}, ...b_1)_2) + \beta_{b_i}$

We evaluate it recursively as in the GJ case and get the **proof** of the **k - representation** closed formula for RF

$$CF: \quad \mathbf{f((1, b_{m-1}, ...b_1, b_0)_2) = (\alpha, \beta_{b_{m-1}}, ...\beta_{b_0})_k}$$

PROOF of the k- Representation CF

We evaluate

$$
\begin{aligned}
f((b_m, b_{m-1}, ..., b_1, b_0)_2) &= k f((b_m, b_{m-1}, ..., b_1)_2) + \beta_{b_0} \\
&= k(k f((b_m, b_{m-1}, ..., b_2)_2) + \beta_{b_1}) + \beta_{b_0} \\
&= k^2 f((b_m, b_{m-1}, ..., b_2)_2) + k\beta_{b_1} + \beta_{b_0} \\
&= k^3 f((b_m, b_{m-1}, ..., b_3)_2) + k^2\beta_{b_2} + k\beta_{b_1} + \beta_{b_0} \\
&\vdots \\
&= k^m f((b_m)_2) + k^{m-1}\beta_{b_{m-1}} + ... + k\beta_{b_1} + \beta_{b_0} \\
&= k^m \alpha + k^{m-1}\beta_{b_{m-1}} + ... + k^2\beta_{b_2} + k\beta_{b_1} + \beta_{b_0} \\
&= (\alpha, \beta_{b_{m-1}}, ..., \beta_{b_1}, \beta_{b_0})_k
\end{aligned}
$$

$$
\begin{aligned}
f((b_m, b_{m-1}, ..., b_1, b_0)_2) &= (\alpha, \beta_{b_{m-1}} .. \beta_{b_1}, \beta_{b_0})_k \\
\text{base 2} &\rightarrow \text{base k}
\end{aligned}
$$

Example

Given RF:

$$\begin{aligned} f(1) &= 5 \\ f(2n) &= 6f(n) + 3 \\ f(2n+1) &= 6f(n) - 10 \end{aligned}$$

Evaluate: $f(100)$ by the use of the k- **representation** and closed formula

$$CF: \quad \mathbf{f((1, b_{m-1}, ...b_1, b_0)_2)} = (\alpha, \beta_{\mathbf{b_{m-1}}}, ...\beta_{\mathbf{b_0}})_{\mathbf{k}}$$

where β_{b_j} are defined as before by

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} \quad ; \quad j = 0, ..., m-1$$

Example Solution

Given

$$\begin{aligned} f(1) &= 5 \\ f(2n) &= 6f(n) + 3 \\ f(2n+1) &= 6f(n) - 10 \end{aligned}$$

We evaluate

$$\alpha = 5$$
$$\beta_0 = 3$$
$$\beta_1 = -10$$

Example Solution

Evaluate: $f(100)$

$$\alpha = 5,\ \beta_0 = 3,\ \beta_1 = -10,\ k = 6,\ n \quad = \quad (\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ )_2$$

$$(b_6 b_5 b_4 b_3 b_2 b_1 b_0)$$

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases}, \qquad j = 0, ..., m-1,$$

$$\beta_{b_0} = 3,\ \ \beta_{b_1} = 3,\ \ \beta_{b_2} = -10,\ \ \beta_{b_3} = 3;\ \ \beta_{b_4} = 3,$$

$$\beta_{b_5} = -10,\ \ \alpha = 5$$

$$\mathbf{f(100) = f((\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ )_2) = (5, -10, 3, 3, -10, 3, 3)_6}$$

More General GJ Function

Further Generalization of GJ

RF:

$$f(i) = \alpha_i, \qquad\qquad i = 1, ..., d-1$$
$$f(dn + j) = cf(n) + \beta_j, \qquad n \geq 1, \ \ 0 \leq j < d$$

We prove the following closed formula

CF:

$$\mathbf{f((b_m, b_{m-1}, ..., b_1, b_0)_d)} \ \ = \ \ (\alpha_{\mathbf{b_m}}, \beta_{\mathbf{b_{m-1}}} ... \beta_{\mathbf{b_1}}, \beta_{\mathbf{b_0}})_{\mathbf{c}}$$

Example

$$\begin{aligned} f(1) &= 34 \\ f(2) &= 5 \\ f(3n) &= 10f(n) + 76 \\ f(3n+1) &= 10f(n) - 2 \\ f(3n+2) &= 10f(n) + 8 \end{aligned}$$

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \\ \beta_2 & b_j = 2 \end{cases}, \qquad j = 0, ..., d-1,$$

Example Solution

We evaluate:

$$
\begin{aligned}
i &= 1, 2 \\
j &= 0, 1, 2 \\
\\
d &= 3 \\
c &= 10 \\
\\
\alpha_1 &= 34 \\
\alpha_2 &= 5 \\
\beta_0 &= 76 \\
\beta_1 &= -2 \\
\beta_2 &= 8
\end{aligned}
$$

Example

Evaluate: $f(19)$

$$19 = (201)_3 = 2 \cdot 3^2 + 0 \cdot 3 + 1$$

$$\alpha_{b_2} = \alpha_2 = 5$$
$$\beta_{b_0} = \beta_0 = 76$$
$$\beta_{b_1} = \beta_1 = -2$$

$$\begin{aligned} f(19) &= f((201)_3) \\ &= (5, 76, -2)_{10} \\ &= 5 \cdot 10^2 + 76 \cdot 10 - 2 \\ &= 500 + 760 - 2 \\ &= 1258 \end{aligned}$$

Short Solution

$$\mathbf{f}((\mathbf{b_m}, \mathbf{b_{m-1}}, \ldots, \mathbf{b_1}, \mathbf{b_0})_{\mathbf{d}}) \quad = \quad (\alpha_{\mathbf{b_m}}, \beta_{\mathbf{b_{m-1}}} \ldots \beta_{\mathbf{b_1}}, \beta_{\mathbf{b_0}})_{\mathbf{c}}$$

Take

$$19 = (2\ 0\ 1)_3$$

Corresponding solution is

$$(\alpha_2,\ \ \beta_0,\ \ \beta_1)_{10}$$

we evaluate $\alpha_2 = 5,\ \ \beta_0 = 76,\ \ \beta_1 = -2$ and get

Solution:

$$(5,\ \ 76,\ \ -2)_{10}$$