Lecture 16

REVIEW for FINAL

Classical DISCRETE MATHEMATICS Problems
Some Discrete Mathematics Problems

PART 1: FINITE and INFINITE SETS
**Finite and Infinite Sets**

**Definition 1**
A set \( A \) is **FINITE** if and only if there is a natural number \( n \in N \) and there is a \( 1 \rightarrow 1 \) function \( f \) that maps the set \( \{1, 2, \ldots, n\} \) onto \( A \), i.e.

\[
\begin{align*}
f : & \quad A \quad \overset{1\rightarrow1,onto}{\longrightarrow} \quad B
\end{align*}
\]

**Definition 2**
A set \( A \) is **INFINITE** if and only if it is **NOT FINITE**
Finite and Infinite Sets

Problem 1
Use the above definitions 1, 2 to prove the following

Fact 1
A set $A$ is INFINITE if and only if it contains a countably infinite subset, i.e. one can define a $1-1$ sequence $\{a_n\}_{n \in \mathbb{N}}$ of some elements of $A$

Proof

Part 1  Proof of the Implication

If $A$ is infinite, then we can define a $1-1$ sequence of elements of $A$
Fact 1 Proof

Let $A$ be infinite
We define the 1-1 sequence of elements of $A$

$$a_1, a_2, \ldots, \ldots, \ldots, a_n, \ldots$$

as follows
Observe that $A \neq \emptyset$, because if $A = \emptyset$, the set $A$ would be finite. **Contradiction**
So there is an element of $a \in A$ and we define

$$a_1 = a$$

Consider now a set $A - \{a_1\} = A_1$. $A_1 \neq \emptyset$ because if $A_1 = \emptyset$, then $A - \{a_1\} = \emptyset$ and $A$ would be finite. **Contradiction**
So there is an element $a_2 \in A - \{a_1\}$ and $a_1 \neq a_2$ and we define

$$a_1, a_2$$
Fact 1 Proof

Assume that we defined a set

$$A_n = A - \{a_1, \ldots, a_n\}$$

The set $A_n \neq \emptyset$ because if $A - \{a_1, \ldots, a_n\} = \emptyset$, then $A$ is finite. **Contradiction**

So there is an element

$$a_{n+1} \in A - \{a_1, \ldots, a_n\} \quad \text{and} \quad a_{n+1} \neq a_n \ldots \neq a_1$$

By mathematical induction, we have defined a 1-1 sequence

$$a_1, a_2, \ldots, \ldots, a_n, \ldots$$

of elements of $A$

This ends the proof the Part 1
Fact 1 Proof

Part 2  Proof of the Implication

If the set $A$ contains a 1-1 sequence $a_1, a_2, \ldots, a_n, \ldots$, then $A$ is INFINITE

Assume $A$ is NOT INFINITE; i.e by the Definition 1 $A$ is finite. Every subset of finite set is finite, so we can’t have a 1-1 infinite sequence of elements of $A$. Contradiction
Finite and Infinite Sets - Problem 2

**Problem 2**

Use the **Fact 1** from **Problem 1** to prove the following

**Dedekind Theorem**

For any set $A$,

$A$ is **INFINITE** if and only if there is a proper subset $B$ of the set $A$, such that $|A| = |B|$.

Dedekind Theorem is sometimes used as a definition of the infinite set.
Dedekind Theorem

For any set $A$,

$A$ is INFINITE if and only if there is a proper subset $B$ of the set $A$, such that $|A| = |B|$

Proof

Part 1  Proof of the Implication

*If $A$ is an infinite set, then there is a set $B$ and there is a function $f$ such that*

\[ B \subset A \quad \text{and} \quad f : A \xrightarrow{1-1,\text{onto}} B \]
Dedekind Theorem Proof

Let the set $A$ be infinite

By the Fact 1, we have a 1-1 sequence $a_1, a_2, \ldots, a_n, \ldots$ of elements of $A$

We take $B = A - \{a_1\}$. Obviously $B \subset A$ and we define the function $f : A \rightarrow B$ as follows

$$f(a_1) = a_2, \quad f(a_2) = a_3 \quad \ldots \quad f(a_n) = a_{n+1}$$

$$f(a) = a \quad \text{for all other} \quad a \in A$$

Obviously, $f$ is 1-1, onto

Observe: we have many of other choices of the set $B$
Dedekind Theorem Proof

Part 2  Proof of the Implication

*If there is a proper subset \( B \) of the set \( A \), such that \(|A| = |B|\), then the set \( A \) is INFINITE*

Assume that we have \( B \subset A \) and the function \( f \), such that

\[
f : A \xrightarrow{1-1,onto} B
\]

We use Fact 1 to show that is infinite; i.e we do it by constructing a 1-1 sequence \( a_1 \ldots a_n, \ldots \) of elements of \( A \)

We do it as follows

We know that \( B \subset A \), so \( A - B \neq \emptyset \) and there is \( b \in A - B \)

This is our first element of the sequence \( a_1 \ldots a_n, \ldots \)
Dedekind Theorem Proof

Observe that $f : A \xrightarrow{1-1,onto} B$, so $f(b) \in B$ and $b \in A - B$, hence $f(b) \neq b$ and we take $f(b)$ is our second element of the sequence.

We have now,

$$a_1 = b, \quad a_2 = f(b)$$

and $f(b) \neq b, \quad b \in A - B, \quad f(b) \in B$

Take now a new element $ff(b)$

As $f$ is 1-1 and $f(b) \neq b$, we get $ff(b) \neq f(b) \neq b$ and we defined a one- one finite sequence

$$a_1 = b, \quad a_2 = f(b), \quad a_3 = ff(b)$$

We denote $ff(b) = f^2(b)$
Dedekind Theorem Proof

We continue the construction by mathematical induction.
Assume that we have constructed a 1-1 finite sequence

\[ a_1 = b, \ a_2 = f(b), \ a_3 = f^2(b)f^3(b), \ldots, \ f^n(b) \]

Observe that

\[ ff^n(b) = f^{n+1}(b) \neq f^n(b) \] as the function \( f \) is 1-1.

By mathematical induction, we have that the sequence

\[ \{f^n(b)\}_{n \in \mathbb{N}} \]

is a 1-1 sequence of elements of \( A \) and hence by Fact 1 \( A \) is infinite.
Problem 3

Use technique from the proof of Dedekind Theorem to prove the following

Fact 2
For any infinite set $A$ and its finite subset $B$, $|A| = |A - B|$

Proof
$A$ is infinite, then by Fact 1 there is a 1-1 sequence $a_1, a_2, \ldots, a_n, \ldots$ of elements of $A$
Let $|B| = k$
We choose $k$ 1-1 sequences $\{c_n^k\}_{n \in \mathbb{N}}$ of the sequence $\{a_n\}_{n \in \mathbb{N}}$, such that

$$c_n^j \neq c_n^i \text{ for all } j \neq i, \ 1 \leq i, j \leq k \text{ and } n \in \mathbb{N}$$
Fact 2 Proof

Let $B = \{b_1, \ldots, b_k\}$

We construct a function $f : A \rightarrow A - \{b_1, \ldots, b_k\}$ as follows

$$
\begin{align*}
    f(b_1) &= c_1^1, & f(c_1^1) &= c_2^1, & \ldots, & f(c_n^1) &= c_{n+1}^1 \\
    f(b_2) &= c_1^2, & f(c_1^2) &= c_2^2, & \ldots, & f(c_n^2) &= c_{n+1}^2 \\
    & \vdots \\
    f(b_k) &= c_1^k, & f(c_1^k) &= c_2^k, & \ldots, & f(c_n^k) &= c_{n+1}^k \\
    f(a) &= a & \text{for all } a \in A - B
\end{align*}
$$

As all sequences $\{C_n^m\}_{n \in \mathbb{N}, m=1, \ldots, k}$ are 1-1, and different, the function $f$ is 1-1 and obviously ONTO $A - B$
Problem 4

Use technique from the proof of Dedekind Theorem to prove that the interval \([a, b], \ a < b\) of real numbers is infinite and

\[|[a, b]| = |(a, b)|\]

Solution

Use construction presented in the proof of the Fact 2 to construct a function

\[f: \ [a, b] \overset{1-1,onto}{\longrightarrow} (a, b)\]
Problem 5

Problem 5
Prove the following

Fact 3
For any or any cardinal numbers $M, N, K$,

1. $N \leq N$

2. If $N \leq M$ and $M \leq K$, then $N \leq K$

Solution
1. $N \leq N$ means that for any set $A$, we have that $|A| \leq |A|$

It is established for example, by taking $f(a) = a$, for all $a \in A$, as obviously

$$f : A \stackrel{1-1}{\longrightarrow} A$$
Problem 4 Solution

2. If $N \leq M$ and $M \leq K$, then $N \leq K$

Solution

We have sets $A$, $B$, $C$, such that $|A| = N$, $|B| = M$ and $|C| = K$ and we assume that there are functions $f$ and $g$, such that

$$f : A \xrightarrow{1-1} B \quad \text{and} \quad g : B \xrightarrow{1-1} C$$

We have to construct a function $h$, such that

$$h : A \xrightarrow{1-1} C$$

We take as $h$ a composition of $f$ and $g$, i.e. we put for all $a \in A$, $h(a) = g(f(a))$ and $h$ is obviously 1-1.
Problem 5

Use Mathematical Induction to prove the following property of finite posets

**Property 1** Every non-empty finite poset has at least one maximal element

**Proof**

Let \((A, \leq)\) be a finite, not empty poset such that \(A\) has \(n\)-elements, i.e. \(|A| = n\)

We carry the Mathematical Induction over \(n \in \mathbb{N} - \{0\}\)

**Reminder:** an element \(a_0 \in A\) is a maximal element in a poset \((A, \leq)\) if and only if

\[\neg \exists a \in A (a_0 \neq a \cap a_0 \leq a)\]
Inductive Proof

Base case: \( n = 1 \), so \( A = \{a\} \) and \( a \) is maximal (and minimal, and smallest, and largest) in the poset \( (\{a\}, \leq) \)

Inductive step: Assume that any set \( A \) such that \( |A| = n \) has a maximal element;
Denote by \( a_0 \) the maximal element in \( (A, \leq) \)
Let \( B \) be a set with \( n + 1 \) elements; i.e. we can write \( B \) as \( B = A \cup \{b_0\} \) for \( b_0 \notin A \), for some \( A \) with \( n \) elements
Inductive Proof

By **Inductive Assumption** the poset \((A, \leq)\) has a maximal element \(a_0\)

To show that \((B, \leq)\) has a maximal element we need to consider 3 cases.

1. \(b_0 \leq a_0\); in this case \(a_0\) is also a maximal element in \((B, \leq)\)
2. \(a_0 \leq b_0\); in this case \(b_0\) is a new maximal in \((B, \leq)\)
3. \(a_0, b_0\) are not compatible; in this case \(a_0\) remains maximal in \((B, \leq)\)

By Mathematical Induction we have proved that

\[\forall n \in \mathbb{N} - \{0\} (|A| = n \Rightarrow A \text{ has a maximal element})\]
Problem 6

Definition

Let $D$ be a set, let $n \geq 0$ and let $R \subseteq D^{n+1}$ be a $(n + 1)$-ary relation on $D$. Then the subset $B$ of $D$ is said to be **closed under** $R$ if $b_{n+1} \in B$ whenever $(b_1, \ldots, b_n, b_{n+1}) \in R$.

Any property of the form "the set $B$ is closed under relations $R_1, R_2, \ldots, R_m$" is called a **Closure Property** of $B$. 


Prove the following **Closure Property Theorem**

**CP Theorem**

Let $P$ be a closure property defined by relations on a set $D$, and let $A \subseteq D$

Then there is a **unique minimal** set $B$ such that $B \subseteq A$ and $B$ has property $P$

**Proof** Consider the set if all subsets of $D$ that are closed under relations $R_1, R_2, \ldots, R_m$ and that have $A$ as a subset. We call this set $S$
Consider now

\[ S = \{ X \in 2^D : A \subseteq X \text{ and } X \text{ is closed under } R_1, R_2, \ldots, R_m \} \]

We need to show that the poset \( S = (S, \subseteq) \) has a **unique minimal** element \( B \).

Observe that \( S \neq \emptyset \) as \( D \subseteq S \) and \( D \) is trivially closed under \( R_1, R_2, \ldots, R_m \) and by definition \( A \subseteq D \).

Consider then the set \( B \) which is the intersection of all sets in \( S \), i.e.

\[ B = \bigcap S \]

Obviously \( A \subseteq B \) and we have to show now that \( B \) is closed under all \( R_i \).
CP Theorem Proof

Suppose that $a_1, a_2, \ldots a_{n-1} \in B$, and $a_1, a_2, \ldots a_{n-1}, a_n \in R_i$

Since $B$ is the intersection of all sets in $S$, we have that $a_1, a_2, \ldots a_{n-1} \in X$, for all $X \in S$

But all sets in $S$ are closed under all $R_i$, they also contain $a_n$

Therefore $a_n \in B$ and hence $B$ is closed under all $R_i$

Moreover, $B$ is minimal, because there can be no proper subset $C$ of $B$, such that $A \subseteq C$ and $C$ is closed under all $R_i$

Because then $C$ would be a member of $S$ and thus $C$ would include $B$