

cse547, math547
DISCRETE MATHEMATICS

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LECTURE 7

CHAPTER 2

SUMS

Part 1: Introduction - Lecture 5

Part 2: Sums and Recurrences (1) - Lecture 5

Part 2: Sums and Recurrences (2) - Lecture 6

Part 3: Multiple Sums (1) - Lecture 7

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Part 5: Infinite Sums- Infinite Series - Lecture 10

CHAPTER 2

SUMS

Part 3: Multiple Sums (1) - Lecture 7

Double Sum

Example 1

Double Sum - Two factors:

$$\begin{aligned}\sum_{1 \leq i, j \leq 3} a_i b_j &= a_1 b_1 + a_1 b_2 + a_1 b_3 \\ &\quad + a_2 b_1 + a_2 b_2 + a_2 b_3 \\ &\quad + a_3 b_1 + a_3 b_2 + a_3 b_3\end{aligned}$$

Question

How can we express $\sum_{1 \leq i, j \leq 3} a_i b_j$ in terms of single sums

$$\sum_i a_i \quad \text{and} \quad \sum_j b_j ?$$

Double Sum Definition

We define for $1 \leq i, j \leq 3$

$$\sum_{1 \leq i, j \leq 3} a_i b_j = \sum_{1 \leq j \leq 3} \left(\sum_{1 \leq i \leq 3} a_i b_j \right) = \sum_{1 \leq j \leq 3} \left(\sum_{1 \leq i \leq 3} a_i b_j \right)$$

General Definition for $i \in I, j \in J$

$$\sum_{i, j} a_i b_j = \sum_i \sum_j a_i b_j = \sum_j \sum_i a_i b_j$$

where we write

$$\sum_{i, j} a_i b_j \quad \text{for} \quad \sum_{i \in I, j \in J} a_i b_j \quad \text{and} \quad \sum_i \sum_j a_i b_j \quad \text{for} \quad \sum_{i \in I} \left(\sum_{j \in J} a_i b_j \right)$$

Example 1

We evaluate the following for $i, j \in \{1, 2, 3\}$

$$\begin{aligned} \sum_{1 \leq i, j \leq 3} a_i b_j &= a_1 b_1 + a_1 b_2 + a_1 b_3 \\ &\quad + a_2 b_1 + a_2 b_2 + a_2 b_3 \\ &\quad + a_3 b_1 + a_3 b_2 + a_3 b_3 \\ &= a_1(b_1 + b_2 + b_3) \quad \text{we pull out} \\ &\quad + a_2(b_1 + b_2 + b_3) \quad \text{the common factor} \\ &\quad + a_3(b_1 + b_2 + b_3) \\ &= (b_1 + b_2 + b_3)(a_1 + a_2 + a_3) \\ &= (a_1 + a_2 + a_3)(b_1 + b_2 + b_3) \end{aligned}$$

Distributive Property

We have **proved** the following property

$$\sum_{1 \leq i \leq 3} \sum_{1 \leq j \leq 3} a_i b_j = \left(\sum_{1 \leq i \leq 3} a_i \right) \left(\sum_{1 \leq j \leq 3} b_j \right) = \left(\sum_{1 \leq j \leq 3} b_j \right) \left(\sum_{1 \leq i \leq 3} a_i \right)$$

Distributive Property for $1 \leq i, j \leq 3$

$$\sum_{i,j} a_i b_j = \left(\sum_i a_i \right) \left(\sum_j b_j \right)$$

Can we **generalize** it?

General Distributive Law

Now **our goal** is to prove the following

General Distributive Law

$$\sum_{i \in I, j \in J} a_i b_j = \left(\sum_{i \in I} a_i \right) \left(\sum_{j \in J} b_j \right)$$

In order to do so we need to bring in our **notation** and general definitions

We write

$$\sum_{i \in I} a_i = \sum_{P(i)} a_i = \sum_{i \in I} a_i [P(i)]$$

where

$I = \{i : P(i)\}$ \longrightarrow $P(i)$ is a predicate defining set I
and $[P(x)]$ is a **characteristic function** of $P(i)$

$$[P(x)] = \begin{cases} 1 & P(x) \text{ true} \\ 0 & P(x) \text{ false} \end{cases}$$

General Distributive Law

In write in a similar way

$$\sum_{j \in J} b_j = \sum_{Q(j)} b_j = \sum_j b_j[Q(j)]$$

where $J = \{j : Q(j)\}$ and $Q(j)$ is a predicate defining set J of indices

We re-write the **General Distributive Law** as follows

$$\sum_{i \in I, j \in J} a_i b_j = \left(\sum_i a_i[P(i)] \right) \left(\sum_j b_j[Q(j)] \right)$$

Question : HOW TO RELATE LEFT SIDE TO RIGHT SIDE ?

Back top Example 1

Let's go back to our **Example 1**

We proved **Distributivity Property** for $1 \leq i, j \leq 3$

$$\sum_{1 \leq i, j \leq 3} a_i b_j = \left(\sum_{1 \leq i \leq 3} a_i \right) \left(\sum_{1 \leq j \leq 3} b_j \right)$$

Observe that we have here the following predicated defining the sets of indexes

$$P(i, j) = (1 \leq i, j \leq 3) = (1 \leq i \leq 3) \cap (1 \leq j \leq 3)$$

$$P_1(i) = (1 \leq i \leq 3) \quad \text{and} \quad P_2(j) = (1 \leq j \leq 3)$$

Hence

$$P(i, j) = P_1(i) \cap P_2(j)$$

General Distributive Law

By definition

$$\sum_{P(i,j)} a_i b_j = \sum_{P_1(i)} \sum_{P_2(j)} a_i b_j$$

when

$$P(i, j) = P_1(i) \cap P_2(j)$$

We want to prove the the following form of the
General Distributive Law

$$\sum_{P(i,j)} a_i b_j = \left(\sum_{P_1(i)} a_i \right) \left(\sum_{P_2(j)} b_j \right)$$

Distributive Law

Let $P(i, j) = P_1(i) \cap P_2(j)$

Observe that

$$[P_1(i) \cap P_2(j)] = [P_1(i)][P_2(j)]$$

Prove it as an exercise;

This is true for any **characteristic functions**

We use this fact and definitions in our calculations on the next slide

Proof of the Distributive Law

$$\begin{aligned}\sum_{P(i,j)} a_i b_j &= \sum_{i,j} a_i b_j [P(i,j)] \\ &= \sum_{i,j} a_i b_j [P_1(i)][P_2(j)] \\ &= \sum_i \left(\sum_j a_i b_j [P_1(i)][P_2(j)] \right)\end{aligned}$$

pull out $a_i [P_1(i)]$ independent on j

$$= \sum_i (a_i [P_1(i)] \sum_j b_j [P_2(j)])$$

Proof of the Distributive Law

We have that

$$\sum_{P(i,j)} a_i b_j = \sum_i (a_i [P_1(i)] \sum_j b_j [P_2(j)])$$

pull out $\sum_j b_j [P_2(j)]$ independent on i

$$= \left(\sum_j b_j [P_2(j)] \right) \left(\sum_i a_i [P_1(i)] \right)$$

$$= \left(\sum_{P_1(i)} a_i \right) \left(\sum_{P_2(j)} b_j \right)$$

end of the **proof**

Distributive Law

We have **proved** our **General Distributive Law**

$$\sum_{P(i, j)} a_i b_j = \sum_{P_1(i) \cap P_2(j)} a_i b_j = \left(\sum_{P_1(i)} a_i \right) \left(\sum_{P_2(j)} b_j \right)$$

also written as

$$\sum_{i \in I, j \in J} a_i b_j = \left(\sum_{i \in I} a_i \right) \left(\sum_{j \in J} b_j \right)$$

Distributive Law Example

Example of application of the **Distributive Law**

$$\sum_{i \in I, j \in J} a_i b_j = \left(\sum_{i \in I} a_i \right) \left(\sum_{j \in J} b_j \right)$$

Consider the following array ($n \times n$)

$$A = \begin{bmatrix} a_1 a_1 & a_1 a_2 \dots & a_1 a_n \\ a_2 a_1 & a_2 a_2 \dots & a_2 a_n \\ \vdots & & \\ a_n a_1 & a_n a_2 \dots & a_n a_n \end{bmatrix}$$

we have here $a_i = a_i$ $b_j = a_j$, where a_i, b_j denote sequences in the **Distributive Law**

Goal : Find

$$\sum_{i,j} a_i a_j$$

Distributive Law Example

Sub-Goal : Find a simple formula for **sum** of all elements **above** or **on** **main diagonal**

$$S_{\nabla} = \sum_{1 \leq i \leq j \leq n} a_i a_j$$

OBSERVATION 1

$$a_i a_j = a_j a_i$$

for any i, j

We denote

$$S_{\Delta} = \sum_{1 \leq j \leq i \leq n} a_i a_j$$

sum of all elements **below** or **on** **main diagonal**

Distributive Law Example

We will now **prove** that

$$S_{\nabla} = S_{\Delta}$$

We now evaluate

$$S_{\nabla} = \sum_{1 \leq i \leq n, 1 \leq j \leq n, i \leq j} a_i a_j = \sum_{P(i,j), i \leq j} a_i a_j$$

for

$$P(i,j) = (1 \leq i \leq n) \cap (1 \leq j \leq n) = Q(i) \cap Q(j) = P(j,i)$$

Distributive Law Example

S_{Δ} becomes now

$$\boxed{S_{\Delta}} = \sum_{1 \leq i \leq n, 1 \leq j \leq n, j \leq i} a_i a_j = \boxed{\sum_{P(i,j), j \leq i} a_i a_j}$$

We evaluate on the next slide

Distributive Law Example

USE: $P(i, j) = P(j, i)$

$$S_{\nabla} = \sum_{P(i,j), i \leq j} a_i a_j = \boxed{\sum_{P(j,i), j \leq i} a_j a_i} = \sum_{P(i,j), j \leq i} a_i a_j = S_{\Delta}$$

↓ ↓↑

Re-name $j \rightarrow i, i \rightarrow j$

We proved

$$S_{\nabla} = S_{\Delta}$$

Distributive Law Example

EVALUATE (remember: our GOAL is to FIND S_{Δ})

$$2S_{\nabla} = S_{\nabla} + S_{\Delta}$$

$$= \sum_{P(i,j), i \leq j} a_i a_j + \sum_{P(i,j), j \leq i} a_i a_j$$


$$Q(i,j)$$

$$R(i,j)$$

$$= \sum_{Q(i,j)} a_i a_j + \sum_{R(i,j)} a_i a_j$$

WE WANT NOW TO **COMBINE DOMAINS** $Q(i,j)$ and $R(i,j)$

Combining Domains

Formula for **COMBINED DOMAINS**

$$\sum_{Q(k)} a_k + \sum_{R(k)} a_k = \sum_{Q(k) \cap R(k)} a_k + \sum_{Q(k) \cup R(k)} a_k$$

OR

$$\sum_{k \in K} a_k + \sum_{k \in K'} a_k = \sum_{k \in K \cap K'} a_k + \sum_{k \in K \cup K'} a_k$$

The second formula is listed **without the proof** on page 31 in our BOOK

Combined Domains Property

Exercise

Prove using the **Truth Tables** and definition of the characteristic function that the following holds

Combined Domains Property

For any predicates $P(k)$, $Q(k)$

$$[P(k) \cup Q(k)] = [P(k)] + [Q(k)] - [P(k) \cap Q(k)]$$

Combined Domains Proof

Proof

We evaluate

$$\begin{aligned} \sum_{Q(k) \cup R(k)} a_k &= \sum_k a_k [Q(k) \cup R(k)] \\ &= \sum_k a_k ([Q(k)] + [R(k)] - [Q(k) \cap R(k)]) \\ &= \sum_k a_k [Q(k)] + \sum_k a_k [R(k)] - \sum_k a_k [Q(k) \cap R(k)] \\ &= \sum_{Q(k)} a_k + \sum_{R(k)} a_k - \sum_{Q(k) \cap R(k)} a_k \end{aligned}$$

Back to Combined Domains Example

REMINDER

$$\boxed{S_{\nabla}} = \sum_{1 \leq i \leq n, 1 \leq j \leq n, i \leq j} a_i a_j = \boxed{\sum_{P(i,j), i \leq j} a_i a_j}$$

for

$$P(i,j) = (1 \leq i \leq n) \cap (1 \leq j \leq n) = Q(i) \cap Q(j)$$

$$\boxed{S_{\Delta}} = \sum_{1 \leq i \leq n, 1 \leq j \leq n, j \leq i} a_i a_j = \boxed{\sum_{P(i,j), j \leq i} a_i a_j}$$

Distributivity Law Example

Our **goal** is to **find** S_{Δ}

$$2S_{\nabla} = S_{\nabla} + S_{\Delta}$$

$$= \sum_{P(i,j), i \leq j} a_i a_j + \sum_{P(i,j), j \leq i} a_i a_j$$



$$Q = Q(i,j) \quad R = R(i,j)$$

$$= \sum_Q a_i a_j + \sum_R a_i a_j$$

Now we **know** how to **COMBINE DOMAINS** $Q(i,j)$ and $R(i,j)$

Distributivity Law Example

$$\boxed{2S_{\nabla}} = S_{\nabla} + S_{\Delta}$$

$$= \sum_Q a_i a_j + \sum_R a_i a_j$$

$$\boxed{= \sum_{Q \cap R} a_i a_j + \sum_{Q \cup R} a_i a_j}$$

We have to **evaluate** $Q \cap R$ and $Q \cup R$

Distributivity Law Example

We know that $Q = P(i,j) \cap (i \leq j)$ and $R = P(i,j) \cap (j \leq i)$

We now **evaluate** $Q \cap R$ and $Q \cup R$ as follows

$$\begin{aligned} Q \cap R &= (P(i,j) \cap (i \leq j)) \cap (P(i,j) \cap (j \leq i)) \\ &= P(i,j) \cap P(i,j) \cap (i \leq j) \cap (j \leq i) = \boxed{P(i,j) \cap (i = j)} \end{aligned}$$

$$\begin{aligned} Q \cup R &= (P(i,j) \cap (i \leq j)) \cup (P(i,j) \cap (j \leq i)) \\ &= P(i,j) \cap ((i \leq j) \cup (j \leq i)) = P(i,j) \cap \text{True} = \boxed{P(i,j)} \end{aligned}$$

Distributivity Law Example

Reminder: $P(i, j) = 1 \leq i \leq n \cap 1 \leq j \leq n$ and we **put it all together** as follows

$$\begin{aligned} 2S_{\nabla} &= \sum_{Q \cap R} a_i a_j + \sum_{Q \cup R} a_i a_j \\ &= \sum_{1 \leq i \leq n, 1 \leq j \leq n} a_i a_j + \sum_{1 \leq i \leq n, 1 \leq j \leq n, i=j} a_i a_j \\ &= \text{DLaw} \left(\sum_{1 \leq i \leq n} a_i \right) \left(\sum_{1 \leq j \leq n} a_j \right) + \left(\sum_{1 \leq i \leq n} a_i^2 \right) \end{aligned}$$

Distributivity Law Example

$$2S_{\nabla} \stackrel{DLaw}{=} \left(\sum_{1 \leq i \leq n} a_i \right) \left(\sum_{1 \leq j \leq n} a_j \right) + \left(\sum_{1 \leq i \leq n} a_i^2 \right)$$

we rename $j \rightarrow i$

$$= \left(\sum_{1 \leq i \leq n} a_i \right)^2 + \left(\sum_{1 \leq i \leq n} a_i^2 \right)$$

Finally, we get:

$$S_{\nabla} = \frac{1}{2} \left(\left(\sum_{1 \leq i \leq n} a_i \right)^2 + \left(\sum_{1 \leq i \leq n} a_i^2 \right) \right)$$

S_{∇} Short Solution

Find $S_{\nabla} = \sum_{1 \leq i, j \leq n} a_i a_j$

Step1: EVALUATE

$$S = \sum_{1 \leq i \leq n, 1 \leq j \leq n} a_i a_j =^{DLaw} \left(\sum_{1 \leq i \leq n} a_i \right) \left(\sum_{1 \leq j \leq n} a_j \right)$$

Step 2 PROVE : $S_{\nabla} = S_{\Delta}$

S_{∇} Short Solution

Step 3 OBSERVE:

$$S = S_{\nabla} + S_{\Delta} - \sum_{1 \leq j \leq n} (a_j)^2 = 2S_{\nabla} - \sum_{1 \leq i \leq n} (a_i)^2$$

Solve on S_{∇}

$$S_{\nabla} = \frac{1}{2} \left(\left(\sum_{1 \leq i \leq n} a_i \right)^2 + \left(\sum_{1 \leq i \leq n} a_i^2 \right) \right)$$

New Problem

Given sequences $\{a_n\}_{n \in \mathbb{N}}$, $\{b_n\}_{n \in \mathbb{N}}$

EVALUATE the SUM

$$S = \sum_{1 \leq j < k \leq n} (a_k - a_j)(b_k - b_j)$$

Observe that

$$1 \leq j < k \leq n = (1 \leq j \leq n) \cap (1 \leq k \leq n) \cap (j < k)$$

denote $P(j, k) = (1 \leq j \leq n) \cap (1 \leq k \leq n)$ observe that
 $P(j, k) = P(k, j)$

and we re-write the limits of our S as follows

$$1 \leq j < k \leq n = P(j, k) \cap (j < k) = P(k, j) \cap (j < k)$$

New Problem

We write now out SUM as

Equation 1

$$S = \sum_{P(j,k), j < k} (a_k - a_j)(b_k - b_j)$$

Now we EXCHANGE j and k (re-name) in S and use $P(j, k) = P(k, j)$ and we get

$$S = \sum_{P(j,k), k < j} (a_j - a_k)(b_j - b_k)$$

New Problem

We evaluate

$$(a_j - a_k)(b_j - b_k) = -(a_k - a_j)(-(b_k - b_j)) = (a_k - a_j)(b_k - b_j)$$

and S becomes now

Equation 2

$$S = \sum_{P(j,k), k < j} (a_k - a_j)(b_k - b_j)$$

We ADD now **Equation 1** and **Equation 2** and get

Equation 3

$$2S = \sum_{P(j,k), j \leq k} (a_k - a_j)(b_k - b_j) + \sum_{P(j,k), k \leq j} (a_k - a_j)(b_k - b_j)$$

Observations

Observation 1 We could change the original limits of summation in both sums from $j < k, k < j$ to $j \leq k, k \leq j$, respectively because the condition $k = j$ gives in both sum the term equal 0, i.e we have $(a_k - a_k) = (b_k - b_k) = 0$

Observation 2 To evaluate 2S we need to use the formula for **combining domains**

$$\sum_Q a_k + \sum_R a_k = \sum_{Q \cap R} a_k + \sum_{Q \cup R} a_k$$

for $Q = P(j, k) \cap (j \leq k)$ and $R = P(j, k) \cap (k \leq j)$

Observations

EVALUATE $Q \cap R$

$$Q \cap R = P(j, k) \cap (j \leq k) \cap P(j, k) \cap (k \leq j) = P(j, k) \cap (k = j)$$

EVALUATE $Q \cup R$

$$\begin{aligned} Q \cup R &= (P(j, k) \cap (j \leq k)) \cup (P(j, k) \cap (k \leq j)) \\ &= P(j, k) \cap (j \leq k \cup k \leq j) = P(j, k) \cap \text{True} = P(j, k) \end{aligned}$$

We re-write our **2S** from **Equation 3** as follows

Back to the Problem

$$\begin{aligned} 2S &= \sum_{P(j,k), j \leq k} (a_k - a_j)(b_k - b_j) + \sum_{P(j,k), k \leq j} (a_k - a_j)(b_k - b_j) \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \\ &\quad \mathbf{Q} \qquad \qquad \qquad \mathbf{R} \\ &= \sum_{Q \cap R} (a_k - a_j)(b_k - b_j) + \sum_{Q \cup R} (a_k - a_j)(b_k - b_j) \end{aligned}$$

We have that $Q \cap R = P(j,k) \cap (k=j)$, $Q \cup R = P(j,k)$, and

$$\sum_{P(j,k) \cap (k=j)} (a_k - a_j)(b_k - b_j) = 0, \text{ so}$$

$$2S = \sum_{P(j,k)} (a_k - a_j)(b_k - b_j)$$

Back to the Problem

Let's expand now

$$(a_k - a_j)(b_k - b_j) = a_k b_k - a_j b_k - a_k b_j + a_j b_j$$

Back to our sum

$$2S = \sum_{P(j,k)} a_k b_k + \sum_{P(j,k)} a_j b_j - 2 \sum_{P(j,k)} a_k b_j$$

re-name $j \rightarrow k$, in second sum and get

$$2S = 2 \sum_{P(j,k)} a_k b_k - 2 \sum_{P(j,k)} a_k b_j \quad \text{and so}$$

$$S = \sum_{P(j,k)} a_k b_k - \sum_{P(j,k)} a_k b_j$$

Back to the Problem

Use distributivity for for the second sum

$$\begin{aligned}\sum_{P(j,k)} a_k b_j &= \sum_{1 \leq j \leq n, 1 \leq k \leq n} a_k b_j \\ &= \left(\sum_{1 \leq k \leq n} a_k \right) \left(\sum_{1 \leq j \leq n} b_j \right) \quad \text{re-name } j \rightarrow k \\ &= \left(\sum_{1 \leq k \leq n} a_k \right) \left(\sum_{1 \leq k \leq n} b_k \right)\end{aligned}$$

Back to the Problem

We evaluate the first sum $\sum_{P(j,k)} a_k b_k$ separately

$$\sum_{P(j,k)} a_k b_k \stackrel{\text{def}}{=} \sum_{1 \leq j \leq n, 1 \leq k \leq n} a_k b_k$$

$$\stackrel{\text{def}}{=} \sum_{1 \leq k \leq n} \left(\sum_{1 \leq j \leq n} a_k b_k \right)$$

↑

$a_k b_k$ constant with respect to j

$$= \sum_{1 \leq k \leq n} (a_k b_k \sum_{1 \leq j \leq n} 1)$$

$$= \sum_{1 \leq k \leq n} a_k b_k n \quad \leftarrow \quad n \text{ is constant with respect to } k$$

$$= n \sum_{1 \leq k \leq n} a_k b_k$$

Solution

We put evaluated components into

$$S = \sum_{P(j,k)} a_k b_k - \sum_{P(j,k)} a_k b_j$$

and get

$$S = n \sum_{1 \leq k \leq n} a_k b_k - \left(\sum_{1 \leq k \leq n} a_k \right) \left(\sum_{1 \leq k \leq n} b_k \right)$$

FORMULA Multiple Sum \rightarrow SingleSums is

$$\sum_{1 \leq j < k \leq n} (a_k - a_j)(b_k - b_j) = n \sum_{1 \leq k \leq n} a_k b_k - \left(\sum_{1 \leq k \leq n} a_k \right) \left(\sum_{1 \leq k \leq n} b_k \right)$$

Formula Application

We use the **FORMULA** to evaluate relationships between

$$\sum_{1 \leq k \leq n} a_k b_k$$

and

$$\left(\sum_{1 \leq k \leq n} a_k \right) \left(\sum_{1 \leq k \leq n} b_k \right)$$

Obtained relationships are called

CHEBYSHEV'S INEQUALITIES

Chebyshev's Inequalities

We re-write the **FORMULA** as follows

$$\left(\sum_{1 \leq k \leq n} a_k \right) \left(\sum_{1 \leq k \leq n} b_k \right) = n \sum_{1 \leq k \leq n} a_k b_k - \left(\sum_{P(k,j), j < k} (a_k - a_j)(b_k - b_j) \right)$$

ASSUME

C1:

$$a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n$$

$$b_1 \leq b_2 \leq b_3 \leq \dots \leq b_n$$

Chebyshev's Inequalities

Observe that under the condition C1

$(a_k - a_j)$, $(b_k - b_j)$ and hence the sum

$\sum (a_k - a_j)(b_k - b_j)$ are all **POSITIVE** for $j < k$

Hence we get that

Chebyshev Inequality 1

$$\left(\sum_{1 \leq k \leq n} a_k \right) \left(\sum_{1 \leq k \leq n} b_k \right) \leq n \sum_{1 \leq k \leq n} a_k b_k$$

holds for

$$a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n$$

$$b_1 \leq b_2 \leq b_3 \leq \dots \leq b_n$$

Chebyshev's Inequalities

Assume now conditions C2:

$$a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n$$

$$b_1 \geq b_2 \geq b_3 \geq \dots \geq b_n$$

Observe that under the conditions C2

$\sum (a_k - a_j)(b_k - b_j)$ is **NEGATIVE** as it has all negative terms and hence

$-\sum (a_k - a_j)(b_k - b_j)$ is **POSITIVE** and we get

Chebyshev Inequality 2

$$\left(\sum_{1 \leq k \leq n} a_k \right) \left(\sum_{1 \leq k \leq n} b_k \right) \geq n \sum_{1 \leq k \leq n} a_k b_k$$