cse547, math547
DISCRETE MATHEMATICS

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LECTURE 6

CHAPTER 2
SUMS

Part 2: Sums and Recurrences

Certain Type of Recurrence

We present now a general technique for finding a CF formula for any Recurrence of a Type:

$$\boxed{\text{RF:} \quad a_n T_n = b_n T_{n-1} + c_n} \quad \text{for} \quad n \geq 1$$

with some Initial Condition for $n = 0$.

where a_n, b_n, c_n are any sequences, $n \geq 1$

We do it by by reducing our RF to a certain sum

Idea: multiply RF by a Summation Factor s_n, $n \geq 1$

We don't know yet what this factor is, but we will find it out

General Technique

Given the general function

$$\boxed{a_n T_n = b_n T_{n-1} + c_n \quad \text{for } n \geq 1} \leftarrow \text{RF}$$

We multiply both sides by s_n, called a Summation Factor and get

$$s_n a_n T_n = s_n b_n T_{n-1} + s_n c_n \qquad \boxed{\star}$$

We want s_n to have a property

$$s_n b_n = s_{n-1} a_{n-1} \qquad \boxed{\text{P}}$$

General Technique

Replacing $s_n b_n$ of $\boxed{\star}$ with corresponding factor defined by $\boxed{P}$ i.e. by $s_{n-1}a_{n-1}$ we get

$$s_n a_n T_n = s_{n-1}a_{n-1}T_{n-1} + s_n c_n \qquad \boxed{\star\star}$$

We put now

$$S_n = s_n a_n T_n \qquad \boxed{S}$$

We use $\boxed{S}$ to re-write $\boxed{\star\star}$ and get

$$S_n = S_{n-1} + s_n c_n \quad \text{for} \quad n \geq 1$$

General Technique

We just developed formula

$$S_n = S_{n-1} + s_n c_n \quad \text{for} \quad n \geq 1$$

Let's evaluate its few terms

$$S_1 = S_0 + s_1 c_1$$

$$S_2 = S_1 + s_2 c_2 = s_0 + s_1 c_1 + s_2 c_2$$

$$S_3 = S_2 + s_3 c_3 = S_0 + s_1 c_1 + s_2 c_2 + s_3 c_3$$

$$S_3 = S_0 + \sum_{k=1}^{3} s_k c_k$$

General Technique

We generalize S_3 (proof by mathematical induction)

$$\boxed{S_n = S_0 + \sum_{k=1}^{n} s_k c_k}$$

(s_k is summation factor)

We now use [S]: $S_n = s_n a_n T_n$

When $n = 0$ we get $S_0 = s_0 a_0 T_0$ and

$$S_n = s_0 a_0 T_0 + \sum_{k=1}^{n} s_k c_k$$

Using [P]: $s_{n-1} a_{n-1} = s_n b_n$ we get

$$\boxed{S_n = s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k}$$

General Technique

We just proved that

$$S_n = s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k$$

By $\boxed{S}$: $\quad S_n = s_n a_n T_n \quad$ we get

$$T_n = \frac{S_n}{a_n s_n} \quad \text{i. e.} \quad T_n = \frac{1}{a_n s_n} S_n$$

Finally we get the following "SUM" closed formula for T_n

$$\boxed{T_n = \frac{1}{a_n s_n} \left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k\right)}$$

Summation Factor

Next Step: Find the summation factor s_n in terms of a_n, b_n, c_n

Question: How to do it??

Answer: Use $\boxed{P}$: $s_{n-1}a_{n-1} = s_n b_n$

Remember that the sequences $(a_n, b_n$ are given for or $n \geq 1$

We evaluate

$$s_2 = \frac{s_1 a_1}{b_2} = s_1 \frac{a_1}{b_2}$$

$$s_3 = \frac{s_2 a_2}{b_3} = s_1 \frac{a_1 a_2}{b_2 b_3}$$

$$s_4 = \frac{s_3 a_3}{b_4} = s_1 \frac{a_1 a_2 a_3}{b_2 b_3 b_4}$$

Summation Factor

We guess and prove by Mathematical Induction that **Summation Factor** is:

$$\boxed{s_n = s_1 \frac{a_1 a_2 \ldots\ldots a_{n-1}}{b_2 b_3 b_4 \ldots\ldots b_n}} \leftarrow \quad \text{where } s_1 \text{ is a constant}$$

Now we put all together and get CF formula for any Recurrence of the Type:

$$\boxed{a_n T_n = b_n T_{n-1} + c_n} \quad \leftarrow \quad \text{RF} \quad \text{for } n \geq 1$$

and where T_0 is given by initial condition

CF for RF

Let RF be any Recurrence of the Type:

$$\boxed{a_n T_n = b_n T_{n-1} + c_n} \quad \leftarrow \quad \text{RF} \quad \text{for } n \geq 1$$

It always have a "sum" CF Formula

$$\boxed{T_n \;=\; \frac{1}{a_n s_n}\,(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k)} \quad \leftarrow \quad \text{CF}$$

where the summation factor s_k is given by

$$\boxed{s_n = s_1 \frac{a_1 a_2 \ldots\ldots a_{n-1}}{b_2 b_3 b_4 \ldots\ldots b_n}} \quad \leftarrow \quad \text{where } s_1 \text{ is a constant}$$

Example of Tower of Hanoi Revisited Again

Let's look at

$T_0 = 0, \quad T_n = 2T_{n-1} + 1 \quad \text{for } n \geq 1$

as particular case of our general formula

$a_n T_n \;=\; b_n T_{n-1} + c_n \quad \text{for } n \geq 1$

We have in this case $\quad a_n = 1, \;\; b_n = 2, \;\; c_n = 1 \quad$ and $\; s_1 = \frac{1}{2}$

We evaluate the summation factor

$$s_n \;=\; \underbrace{\frac{1}{2...2}}_{n-1}\underbrace{\frac{1}{2}}_{s_1} = \frac{1}{2^n} \qquad s_1 = \frac{1}{2}$$

Therefore, $\quad s_n = 2^{-n}, \qquad s_1 = \frac{1}{2}$

Example of Tower of Hanoi Revisited Again

Check $T_n = \frac{1}{a_n s_n}\left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k\right)$ for

$s_n = 2^{-n}, \quad a_n = 1, \quad b_n = 2, \quad c_n = 1$

So now

$$T_n = \frac{1}{2^{-n}}\left(0 + \sum_{k=1}^{n} \frac{1}{2^n}\right)$$

Observe that $\sum_{k=1}^{n} \frac{1}{2^n}$ is a geometric sum

$S_n = \frac{a_0(q^{n+1}-1)}{q-1}$, for $q = \frac{1}{2} < 1$, so we get

$$\sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n} \quad \text{and} \quad T_n = 2^n\left(1 - \frac{1}{2^n}\right)$$

$\boxed{T_n = 2^n - 1}$ ← CF Formula

Quicksort

Quicksort, Hoare 1962

The number of comparison steps made by the **Quicksort** when applied to n items in random order is given by a function

RF $\quad C_0 = 0, \quad C_n = (n+1) + \frac{2}{n} \sum_{k=0}^{n-1} C_k$

We calculate: $\quad C_1 = 2, \quad C_2 = 5, \quad C_3 = \frac{26}{3}$ etc ...

Goal: find CF for RF

Step 1: Get rid of the $\sum$ in the recurrence

Step 2: Find a CF Formula, or a "sum" CF at least

Hint: use the General Technique

Given RF: $C_n = (n+1) + \frac{2}{n}\sum_{k=0}^{n-1} C_k$

We re-write it as follows

$$nC_n = n^2 + n + 2\sum_{k=0}^{n-1} C_k \quad \text{where } n > 1 \quad \boxed{\star}$$

$$nC_n = n^2 + n + 2\left(\sum_{k=0}^{n-2} C_k + C_{n-1}\right)$$

$$nC_n = n^2 + n + 2\sum_{k=0}^{n-2} C_k + 2C_{n-1} \quad \boxed{1}$$

Quicksort

$$nC_n = n^2 + n + 2\sum_{k=0}^{n-2} C_k + 2C_{n-1} \qquad \boxed{1}$$

We re-write

$$\boxed{\star} \quad nC_n = n^2 + n + 2\sum_{k=0}^{n-1} C_k \quad \text{for} \quad n = n - 1$$

$$(n-1)C_{n-1} = (n-1)^2 + n - 1 + 2\sum_{k=0}^{n-2} C_k$$

$$(n-1)C_{n-1} = n^2 - n + 2\sum_{k=0}^{n-2} C_k \qquad \boxed{2}$$

We subtract $\boxed{2}$ from $\boxed{1}$ and we get

$$nC_n - (n-1)C_{n-1} = 2n + 2C_{n-1} \qquad \boxed{3}$$

Quicksort

$$nC_n = (n-1)C_{n-1} + 2n + 2C_{n-1} \qquad \boxed{3}$$

$$= nC_{n-1} - C_{n-1} + 2n + 2C_{n-1}$$

$$= 2n + nC_{n-1} + C_{n-1}$$

We get the formula

$$RF: \quad nC_n = (n+1)C_{n-1} + 2n \quad \text{and} \quad C_0 = 0$$

This is of the form of the general type

$$a_n T_n = b_n T_{n-1} + c_n$$

for $a_n = n,\quad b_n = n+1,\ c_n = 2n$

Quicksort

We know that the **Summation Factor** multiplied by a constant s_1 is

$$s_n = s_1 \frac{a_1 a_2 .....a_{n-1}}{b_2 b_3 ....b_n}$$

and now $a_n = n, \ b_n = n+1, \ c_n = 2n$

We get

$$s_n = \frac{1 \cdot 2 \cdot \ \dots \ (n-1)}{3 \cdot \ \dots \ (n-1)n(n+1)} = \frac{2}{n(n+1)}$$

as $b_2 = 3$ and $s_1 = \frac{2}{1 \cdot 2} = 1$

Quicksort

Last step: we use formula

$$T_n = \frac{1}{a_n s_n}\left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k\right)$$

for $a_n = n, \; b_n = n+1, \; c_n = 2n$ and get

$$C_n = \frac{1}{n s_n}(0 + \sum_{k=1}^{n} 2k s_k) \qquad (T_0 = C_0 = 0)$$

This gives the following solution for $s_n = \frac{2}{n(n+1)}$

$$C_n = \frac{n(n+1)}{2n} \sum_{k=1}^{n} \frac{4k}{k(k+1)}$$ we pull out 4 out of sum and get

$$''SUM''\,CF: \qquad C_n = 2(n+1)\sum_{k=1}^{n} \frac{1}{k+1}$$

Harmonic Number H_n

$H_n = 1 + \frac{1}{2} + ... + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$, i.e.

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

Name origin: k-th harmonic produced by a violin string is the fundamental tone produced by a string that is $\frac{1}{k}$ times long.

We now use H_n to get a H_n CF formula for our **Quicksort recurrence** " SUM" CF formula

$$\text{" SUM" CF}: \quad C_n = 2(n+1) \sum_{k=1}^{n} \frac{1}{k+1}$$

H_n and Quicksort

Observe that

$$\sum_{k=1}^{n} \frac{1}{k+1} = \sum_{1 \le k \le n} \frac{1}{k+1}$$

We want now to **evaluate** the sum

$$\sum_{1 \le k \le n} \frac{1}{k+1} \quad \text{in terms of} \quad H_n$$

H_n and Quicksort

We put $k = k - 1$ and get

$$\sum_{1 \leq k \leq n} \frac{1}{k+1} = \sum_{1 \leq k-1 \leq n} \frac{1}{k}$$

$$= \sum_{2 \leq k \leq n+1} \frac{1}{k}$$

$$= (\sum_{k=1}^{n} \frac{1}{k}) - \frac{1}{1} + \frac{1}{n+1}$$

H_n and Quicksort

We obtained

$$\boxed{\sum_{k=1}^{n} \frac{1}{k+1} = H_n - \frac{n}{n+1}}$$

and so our " SUM" CF formula

$$C_n = 2(n+1) \sum_{k=1}^{n} \frac{1}{k+1}$$

becomes

$$C_n = 2(n+1)(H_n - \tfrac{n}{n+1}) = 2(n+1)H_n - \tfrac{2n(n+1)}{n+1}$$

$$= 2(n+1)H_n - 2n$$

H_n and Quicksort

We have **proved** the **sum-closed** formula

$$\text{"SUM" } CF: \quad C_n = 2(n+1)\sum_{k=1}^{n} \frac{1}{k+1}$$

has its H_n - **closed** formula

$$H_n CF: \quad C_n = 2(n+1)H_n - 2n, \quad C_0 = 0$$

We evaluate (to check the result!)

$C_0 = 0,\ \ C_1 = 1,\ \ C_2 = 2 \cdot 3 \cdot \frac{3}{2} - 4 = 5,$ etc. .

Perturbation Method

Perturbation Method is a method that often allows us to evaluate a CF form for a certain sums

The **idea** is to start with an **unknown sum** and call it S_n:

$$S_n = \sum_{k=0}^{n} a_k$$

Then we re-write S_{n+1} in **two ways**, by splitting off both its last term a_{n+1} and its first term a_0:

$$S_n + a_{n+1} = a_0 + \sum_{k=1}^{n+1} a_k \quad \text{put k:= k+1}$$

$$= a_0 + \sum_{1 \leq k+1 \leq n+1} a_{k+1} = a_0 + \sum_{0 \leq k \leq n} a_{k+1}$$

$$= a_0 + \sum_{k=0}^{n} a_{k+1}$$

Perturbation Method

We get a formula:

$$S_n + a_{n+1} = a_0 + \sum_{k=0}^{n} a_{k+1}$$

Goal of the Perturbation Method is to **work** on the **last sum** $\sum_{k=0}^{n} a_{k+1}$ and try to express it on terms of S_n

If it **works** and if we get a **multiple** of S_n we **solve** the equation on S_n and obtain the closed formula CF for the **original sum**

If it **does not work** - we look for another method

Example 1

Geometric Sum Revisited

1. $$S_n = \sum_{k=0}^{n} ax^k$$

2. Observe:

$$\sum_{k=0}^{n} ax^{k+1} = x \sum_{k=0}^{n} ax^k$$

We **evaluate** by Perturbation Technique

$$S_n + ax^{n+1} = ax^0 + \sum_{k=0}^{n} ax^{k+1}$$

$$= a + x \sum_{k=0}^{n} ax^k = a + xS_n$$

Example 1

We got the following equation on S_n:

$S_n + ax^{n+1} = a + xS_n$

Solve on S_n

$$S_n = \frac{a(1 - x^{n+1})}{1 - x}$$

and

$$\sum_{k=0}^{n} ax^k = \frac{a(1 - x^{n+1})}{1 - x}$$

Example 2

Evaluate using the Perturbation Method

$$S_n = \sum_{k=0}^{n} k2^k$$

We use the **Perturbation Formula**

Now we have

$$S_n + a_{n+1} = a_0 + \sum_{k=0}^{n} a_{k+1}$$

for $a_0 = 0$ and $a_{n+1} = (n+1)2^{n+1}$

$$S_n + (n+1)2^{n+1} = \sum_{k=0}^{n} (k+1)2^{k+1} = \sum_{k=0}^{n} k2^{k+1} + \sum_{k=0}^{n} 2^{k+1}$$

$$= 2\sum_{k=0}^{n} k2^k + (2^{n+2} - 2) \text{ (geometric sum)}$$

Example 2

We get an equation on S_n

$$S_n + (n+1)2^{n+1} \;=\; 2S_n + 2^{n+2} - 2$$

Solution

$$S_n(1-2) \;= -(n+1)2^{n+1} + 2^{n+2} - 2$$

$$S_n = 2^{n+1}(n+1-2) + 2$$

$$S_n = (n-1)2^{n+1} + 2$$

Hence

$$\sum_{k=0}^{n} k2^k \;=\; (n-1)2^{n+1} + 2$$