cse547, math547
DISCRETE MATHEMATICS

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LECTURE 5

CHAPTER 2
SUMS

Part 1: Introduction

Sequences and Sums of Sequences

Sequences

Definition

A **sequence** of elements of a set A is any function f from the set of natural numbers N into A

$$f : N \longrightarrow A$$

Any $f(n) = a_n$ is called n-th term of the sequence f.

Notations:

$$f = \{a_n\}_{n \in N}, \quad \{a_n\}_{n \in N}, \quad \{a_n\}$$

Sequences Example

Example

We define a sequence f of real numbers R as follows

$$f : N \longrightarrow R$$

Given by a formula

$$f(n) = n + \sqrt{n}$$

We also use a shorthand notation for the sequence f and write

$$a_n = n + \sqrt{n}$$

Sequences Example

We often write $f = \{a_n\}$ in an even shorter and more informal form as

$$a_0 = 0, \quad a_1 = 1 + 1 = 2, \quad a_2 = 2 + \sqrt{2}$$

$$0, \quad 2, \quad 2 + \sqrt{2}, \; 3 + \sqrt{3}, \; ....n + \sqrt{n} \dots$$

Observations

Observation 1: A Sequence is **always** INFINITE (countably infinite) as by **definition**, the **domain** of the sequence (function f) is a set of N of natural numbers

Observation 2: card N = card N-K, for K is any **finite** subset of N, so we can enumerate elements of a sequence by any infinite subset of N

Definition: A set T is called **countably infinite** iff card T= card N, i.e. there is a one to one (1-1) function f that maps N onto T, i.e.

$$f: \quad N \longrightarrow^{1-1, onto} T$$

Observations

Observation 3: We can choose as a SET of INDEXES of a sequence any COUNTABLY infinite set T, not only the set N of natural numbers

In our Book: $T = N - \{0\} = N^+$, i.e we consider sequences that "start" with n = 1

We usually write sequences as

$$a_1, \quad a_2, \quad a_3, \quad ..... \quad a_n, \quad .....$$

$$\{a_n\}_{n \in N^+}$$

Finite Sequences

Definition

A **finite sequence** of elements of a set A is any function f from a finite set K into A

In case when K is a non-empty **finite subset** of natural numbers N we write, for simplicity $K = \{1, 2, ...n\}$ and call n the **length** of the sequence

We write sequence function f as

$$f : \{1, 2, ...n\} \longrightarrow A \quad f(n) = a_n, \quad f = \{a_k\}_{k=1....n}$$

Case n=0: the function f is empty we call it an empty sequence and denote by e

Example

Example 1

Let

$$a_n = \frac{n}{(n-2)(n-5)}$$

Domain of the sequence $f(n) = a_n$ is $N - \{2, 5\}$ and

$$f : N - \{2, 5\} \to R$$

Example 2 Let $T = \{-1, -2, 3, 4\}$

$f(n) = a_n$ for $n \in T$ is now a **finite sequence** with the domain T

FINITE SUMS

In **Chapter 2**, we consider only **finite sums** of consecutive elements of sequences $\{a_n\}$ of rational numbers

Definition

Given a sequence f of rational numbers

$$f : N^+ \longrightarrow R \qquad f(n) = a_n$$

We write a **finite sum** as

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + ... + a_n$$

Sums of elements of sequences

We also use notations:

$$\sum_{k=1}^{n} a_k \;=\; \sum_{1 \leq k \leq n} a_k \;=\; \sum_{k \in \{1, \ldots n\}} a_k$$

$$\sum_{k=1}^{n} a_k \;=\; \sum_{K} a_k$$

for $K = \{1, ..n\}$

Sums of elements of sequences

Given a sequence of numbers:

$$\boxed{f : N^+ \rightarrow R, \qquad f(n) = a_n} \longleftarrow \text{FULL DEFINITION}$$

$$\boxed{a_1 a_2 ... a_n, \quad a_k \in R} \longleftarrow \text{SHORTHAND}$$

We sometimes evaluate a **sum** of some sub-sequence of $\{a_n\}$

Sums of elements of sequences

For example we want to sum-up only each second term of $\{a_n\}$, i.e. $n \in EVEN$

We write in two ways:

1. $$\sum_{1\leq k\leq 2n,\ k\in EVEN} a_k = a_2 + a_4 + \ldots.. + a_{2n}$$

where $\boxed{1 \leq k \leq 2n,\ k \in EVEN} \longleftarrow P(k)$ summation property

2. $$\sum_{k=1}^{n} a_{2k} = a_2 + a_4 + \ldots.. + a_{2n}$$

where $\boxed{a_{2k}} \longleftarrow$ subsequence property

Sums Notations

We use following **notations**

$$\sum_{P(k)} a_k = \sum_{k \in K} a_k = \sum_{K} a_k$$

for $K = \{n \in N : P(n)\}$

and P(n) is a certain formula defining our **restriction** on n

We assume the following

1. The set K is **defined**; i.e. the statement $P(n) = True$ or *False* is **decidable**

2. The set K is **finite** - we consider only **finite sums** at this moment

Example 1

Let P(n) be a **property:** $1 \leq n < 100$ and $n \in ODD$

P(n) is a formula defining all ODD numbers between 1 and 99 (included) and hence

$$K = \{n \in N : P(n)\} = \{n \in ODD : 1 < n \leq 99\} = \{1, 3, 5, ..., 99\}$$

or

$$K = \{1, 3, ...(2n+1)\} \quad \text{for} \quad 0 \leq n \leq 49$$

Example 1

We have that $K = \{1, 3, ...(2n+1)\}$ for $0 \leq n \leq 49$ and by definition of the sum

$$\sum_{P(n)} a_n = \sum_{K} a_k$$ ⟵ PROPERTY

$$= \sum_{n=0}^{49} a_{(2n+1)} = a_1 + a_3 + ... + a_{99}$$ ⟵ subsequence

Example 2

Example 2

Let P(n) be a property: $1 \leq n < 100$

P(n) is now a formula defining natural numbers between 1 and 99 (included), i.e.

$$K = \{n \in N: \ P(n)\} = \{n \in N: 1 < n \leq 99\} = \{1, 2, ..., 99\}$$

In this case

$$\sum_{P(n)} a_n \quad = \quad \sum_{K} a_k \quad = \quad \sum_{k=1}^{99} a_k$$

$$= a_1 + a_2 + a_3 + ... + a_{99}$$

Example 3

Let P(n) be a property: $1 \le n < 100$ and

$$a_n = (2n+1)^2$$

Evaluate: $\sum_{P(n)} a_n$

$K = \{P(n) : 1 \le n < 100\} = \{1, 2, .99\}$ and

$$\sum_{P(n)} (2n+1)^2 = \sum_{k=1}^{99} (2n+1)^2$$

$$= 3^2 + 5^2 + ... + (2*99+1)^2$$

USEFUL NOTATION

Here is our **BOOK NOTATION** (from Kenneth Iverson's programming language APL)

Characteristic Function of the formula P(x)

$$[P(x)] = \begin{cases} 1 & P(x) \ \ true \\ 0 & P(x) \ \ false \end{cases}$$

where $x \in X \neq \emptyset$

Example:

Let P(n) be a property: p is prime number

$$[p \ \ prime] = \begin{cases} 1 & p \ \ is \ \ prime \\ 0 & p \ \ is \ \ not \ \ prime \end{cases}$$

Useful Sum Notation

We write

$$\boxed{\sum_{P(k)} a_k = \sum_k a_k [P(k)]} = \sum_{k \in K} a_k$$

where

$$K = \{k : P(k)\}$$

Useful Sum Notation Example

Example

$$\sum_{p} [p \;\; prime] \;\; [p \leq n] \;\; \frac{1}{p}$$

Observe that now

P(x) is $P_1(x) \cap P_2(x)$

for $P_1(x)$: x is prime

$P_2(x)$: $x \leq n$ for $n \in N$

$P(x)$ says : x is prime and $x \leq n$

Example

$$\sum_p \ [p \ \ prime] \ \ [p \leq n] \ \ \frac{1}{p}$$

$\sum$ means :

we sum $\frac{1}{p}$ over all p that are PRIME and $p \leq n$ for $n \in N$

Case when $n = 0$ - as $0 \in N$

We have that $P(x)$ is **false** as PRIMES are numbers ≥ 2

Book Notations Corrections

Book uses notation $p \leq N$ instead of $p \leq n$,

It is tricky!

N in standard notation denotes the **set of natural numbers**

We write $n \in N$ and we **can't write** $n \leq N$

When you read the book now and later, pay attention

Book also uses: $n \leq K$

This really means that $n \leq k$

In standard notation **CAPITAL LETTERS DENOTE SETS**

Book Notations Corrections

Authors never define a sequence $\{a_n\}$ for $\sum a_k$

They also often state:

$"a_k"$ is defined/not defined for all set of INTEGERS

It means they **admit** sequences and FINITE sequences with indices being Integers- what is OK and the set of Integers is **infinitely countable**

Useful Sum Notation Reminder

$$\sum_{P(k)} a_k = \sum_{k \in K} a_k = \sum_{k} [P(k)] a_k$$

where

$$K = \{k \in Z : P(k)\} \text{ and } K \text{ is **finite**}$$

or

$$K = \{k \in N : P(k)\} \text{ and } K \text{ is **finite**}$$ $\leftarrow$ This is usual case

where N is set of Natural numbers, Z - set of Integers

Part 2: Sums and Recurrences

Some Observations

Observation 1: for any $n \in N$

$$\sum_{k=1}^{n+1} a_k = \sum_{k=1}^{n} a_k + a_{n+1}, \quad \text{and} \quad \sum_{k=1}^{1} a_k = a_1$$

Consider case $n = 0$: the sum is undefined and **we put**

$$\sum_{k=1}^{0} a_k = 0$$

In general we put

$$\boxed{\sum_{k=a}^{b} a_k = 0 \quad \text{when} \quad b < a} \leftarrow \text{DEFINITION}$$

Some Observations

Observation 2: for any $n \in N^+$

$$\sum_{k=0}^{n} a_k = \sum_{k=0}^{n-1} a_{k-1} + a_n$$

Now when $n = 0$ we get $\sum_{k=0}^{0} a_k = a_0$

Reminder:

$$\sum_{k=0}^{-1} a_k = 0$$

Sum Recurrence

We know that for any $n \in N^+$

$$\sum_{k=0}^{n} a_k = \sum_{k=0}^{n-1} a_{k-1} + a_n$$

We denote $S_n = \sum_{k=0}^{n} a_k$

Observe that we have defined a function S

$$\boxed{S : N \longrightarrow R, \quad S(n) = S_n = \sum_{k=0}^{n} a_k} \leftarrow \text{SUM FUNCTION}$$

Sum Recurrence

We re-rewrite $S(n) = S_n = \sum_{k=0}^{n} a_k$ and get a following **recursive formula** for S

$$\boxed{S_0 = a_0, \quad S_n = S_{n-1} + a_n \quad \text{for } n > 0}$$

↑

Sum Recurrence Formula

We will use techniques from **Chapter 1** to evaluate (if possible) **closed** formulas for certain SUMS

Problem

Given a sequence

$f : N \longrightarrow R$, defined by a formula

$$f(n) = a_n \quad \text{for} \quad a_n = a + bn$$

where $a, b \in R$ are constants

Problem

Find a closed formula CF for the following sum

$$S(n) = \sum_{k=0}^{n} a_k = \sum_{k=0}^{n} (a + bk)$$

Sum Recurrence

The recurrence form of our sum S_n is

RF: $S_0 = a$

$$S_n = S_{n-1} + \underbrace{(a + bn)}_{a_n}$$

We want to find a Closed Formula CF for this recurrence formula

Generalization

Let's **generalize** our formula RF to RS as follows

$$RS: \quad R_0 = \alpha$$

$$R_n = R_{n-1} + \beta + \gamma n$$

The previous RF is a case of RS for

$\alpha = a, \beta = a, \gamma = b$

From RS to CF

$RF: \quad R_0 = \alpha, \quad R_n = R_{n-1} + \beta + \gamma n$

Step 1: evaluate few terms

$R_0 = \alpha$

$R_1 = \alpha + \beta + \gamma$

$R_2 = \alpha + \beta + \gamma + \beta + 2\gamma = \alpha + 2\beta + 3\gamma$

$R_3 = \alpha + 2\beta + 3\gamma + \beta + 3\gamma = \alpha + 3\beta + 6\gamma$

From RS to CF

Step 2: Observation - **general formula** for CF

$$\boxed{R_n = A(n)\alpha + B(n)\beta + C(n)\gamma} \leftarrow \text{CF}$$

GOAL: Find $A(n), B(n), C(n)$ and **prove** that RS = CF for

RS $R_0 = \alpha, \quad R_n = R_{n-1} + \beta + \gamma n$

Method: Repertoire Method

Repertoire Function 1

RS $R_0 = \alpha, \quad R_n = R_{n-1} + \beta + \gamma n$

CF $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$

We set the first **repertoire function** as

$$\mathbf{R_n} = \mathbf{1} \text{ for all } n \in N$$

We set $R_n = \mathbf{R_n}$, for all $n \in N$ and
$R_0 = \alpha$, and $\mathbf{R_0} = \mathbf{1}$ so $\alpha = 1$

Repertoire Function 1

RS: $R_0 = \alpha, \quad R_n = R_{n-1} + \beta + \gamma n$

Repertoire function is $R_n = 1$ for all $n \in N$

We set $R_n = \mathbf{R_n}$, for all $n \in N$ and we evaluate

$1 = 1 + \beta + \gamma n$ for all $n \in N$

$0 = \beta + \gamma n$ for all $n \in N$

This is possible only when $\beta = \gamma = 0$

Solution

$$\alpha = 1, \quad \beta = 0, \quad \gamma = 0$$

Equation 1

CF: $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$

We use now the first **repertoire function**

$\mathbf{R_n} = \mathbf{1}$ for all $n \in N$

We set $R_n = \mathbf{R_n}$, for all $n \in N$ and use just evaluated

$\alpha = 1, \ \beta = 0, \ \gamma = 0$

and get our **equation 1:**

1 = A(n) , for all $n \in N$

Fact 1 A(n) = 1, for all $n \in N$

Repertoire Function 2

RS: $R_0 = \alpha, \quad R_n = R_{n-1} + \beta + \gamma n$

We set the second **repertoire function** as

$$\mathbf{R_n} = \mathbf{n} \text{ for all } n \in N$$

We set $R_n = \mathbf{R_n}$, for all $n \in N$ and evaluate

$R_0 = \alpha$, and $R_0 = 0$ by definition, so $\alpha = 0$

Repertoire Function 2

RS $R_0 = \alpha, \quad R_n = R_{n-1} + \beta + \gamma n$

The second **repertoire function** is $R_n = n$ for all $n \in N$

We set $R_n = \mathbf{R_n}$, for all $n \in N$ and we evaluate

$n = (n-1) + \beta + \gamma n,$ for all $n \in N$

$0 = \beta - 1 + \gamma n,$ for all $n \in N$

$1 = \beta + \gamma n,$ for all $n \in N$

This is possible only when $\beta = 1, \ \gamma = 0$

Solution

$$\alpha = 0, \quad \beta = 1, \quad \gamma = 0$$

Equation 2

CF $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$

We use now the second **repertoire function**

$\mathbf{R_n} = \mathbf{n}$ for all $n \in N$

We set $R_n = \mathbf{R_n}$, for all $n \in N$ and use just evaluated

$\alpha = 0, \ \beta = 1, \ \gamma = 0$

and get our **equation 2:**

n = B(n) , for all $n \in N$

Fact 2 B(n) = n, for all $n \in N$

Repertoire Function 3

RS $R_0 = \alpha, \quad R_n = R_{n-1} + \beta + \gamma n$

We set the third **repertoire function** as

$$\mathbf{R_n} = \mathbf{n^2} \quad \text{for all} \quad n \in N$$

We set $R_n = \mathbf{R_n}$, for all $n \in N$ and evaluate

$R_0 = \alpha$, and $R_0 = 0$, so $\alpha = 0$

Repertoire Function 3

RS $R_0 = \alpha, \quad R_n = R_{n-1} + \beta + \gamma n$

Third **repertoire function** is

$\mathbf{R_n} = \mathbf{n^2}$ for all $n \in N$

We set $R_n = \mathbf{R_n}$, for all $n \in N$ and evaluate

$n^2 = (n-1)^2 + \beta + \gamma n,$ for all $n \in N$

$n^2 = n^2 - 2n + 1 + \beta + \gamma n,$ for all $n \in N$

$0 = -2n + 1 + \beta + \gamma n$, for all $n \in N$

$0 = (1+\beta) + n(\gamma - 2),$ for all $n \in N$

This is possible only when $\beta = -1, \ \gamma = 2$

Solution $\alpha = 0, \ \beta = -1, \ \gamma = 2$

Equation 3

CF $\quad R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$

We use now the third **repertoire function**

$\mathbf{R_n} = \mathbf{n^2}$ for all $n \in N$

We set $R_n = \mathbf{R_n}$, for all $n \in N$ and use just evaluated

$\alpha = 0, \ \beta = 1, \ \gamma = 0$

and get our **equation 3:**

$2C(n) - B(n) = n^2$, for all $n \in N$

Fact 3 $\quad 2C(n) - B(n) = n^2$, for all $n \in N$

Repertoire Method System of Equations

We obtained the following system of **3 equations** on A(n), B(n), C(n)

1. A(n) = 1

2. B(n) = n

3. $2C(n) - B(n) = n^2$

We substitute **1.** and **2.** in **3.** we get

$n^2 = -n + 2C(n)$ and $C(n) = \frac{(n^2+n)}{2}$

Solution

$$A(n) = 1, \;\; B(n) = n, \;\; C(n) = \frac{(n^2+n)}{2}$$

CF Solution

We now put the **solution** into the general formula

CF: $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$

and get that the closed formula CF equivalent to

RS : $R_0 = \alpha, \quad R_n = R_{n-1} + \beta + \gamma n$ is

$$\boxed{R_n = \alpha + n\beta + (\frac{n^2+n}{2})\gamma}$$

CF Solution

Let's now go back to original sum

$$S_n = \sum_{k=0}^{n} (a + bk)$$

We have that

$S_n = R_n$, for $\alpha = a$, $\beta = a$, $\gamma = b$ so

$S_n = a + na + (\frac{n^2+n}{2})b = (n+1)a + (\frac{n^2+n}{2})b$

We hence evaluated

$$S_n = \sum_{k=0}^{n} (a + bk) = (n+1)a + \frac{n(n+1)}{2} b$$

Simple Solution

Of course we can do it by a MUCH simpler method

$$\sum_{k=0}^{n}(a+bk) = \sum_{k=0}^{n} a + \sum_{k=0}^{n} bk$$

$$= (n+1)a + b\sum_{k=0}^{n} k$$

$$= (n+1)a + \frac{n(n+1)}{2}b$$

Observe that for a sequence $a_n = a$, for all n we get

$$\sum_{k=0}^{n} a_n = \sum_{k=0}^{n} a = a + \ldots.. + a = (n+1)a$$

Summations Laws

Distributive Law

$$\sum_{k \in K} c a_k = c \sum_{k \in K} a_k$$

Associative Law

$$\sum_{k \in K} (a_k + b_k) = \sum_{k \in K} a_k + \sum_{k \in K} b_k$$

Commutative Law

$$\sum_{k \in K} a_k = \sum_{\prod(k) \in K} a_{\prod(k)}$$

$\prod(k)$ - any permutation of elements of K

K - is a finite subset of Integers

Geometric Sum

Geometric Sequence

Definition

A sequence $f : N \rightarrow R$, $f(n) = a_n$ is **geometric** iff

$$\frac{a_{n+1}}{a_n} = q, \quad \text{for all } n \in N$$

We prove a following property of a geometric sequence $\{a_n\}$

$$\boxed{a_n = a_0 q^n} \quad \text{for all } n \in N$$

Geometric Sum Formula

$$\boxed{S_n = \sum_{k=0}^{n} a_0 q^k = \frac{a_0(1-q^{n+1})}{1-q}}$$

Proof of Geometric Sum Formula

$$S_n = \sum_{k=0}^{n} a_0 q^k$$

$$S_n = a_0 + a_0 q + ... + a_0 q^n$$

$$qS_n = a_0 q + a_0 q^2 + ... + a_0 q^n + a_0 q^{n+1}$$

$$S_n(1-q) = a_0 - a_0 q^{n+1}$$

$$\boxed{S_n = \sum_{k=0}^{n} a_0 q^n = \frac{a_0(q^{n+1}-1)}{q-1}} \leftarrow \text{Geometric Sum}$$

Examples

Example 1

$$S_n = \sum_{k=0}^{n} 2^{-k} = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k$$

We have $a_0 = 1, \quad q = \frac{1}{2},$ and

$$S_n = \frac{\left(\frac{1}{2}\right)^{n+1} - 1}{\frac{-1}{2}} = 2 - \left(\frac{1}{2}\right)^n$$

Examples

Example 2

$$S_n = \sum_{k=1}^{n} 2^{-k} = \sum_{k=1}^{n} (\frac{1}{2})^k$$

We have now $a_1 = \frac{1}{2}$, $q = \frac{1}{2}$ and hence $n := n - 1$ and

$$S_{n-1} = \frac{\frac{1}{2}((\frac{1}{2}^n) - 1)}{\frac{-1}{2}} = 1 - (\frac{1}{2})^n$$

From RF to Sum S_n to CF

Tower of Hanoi

RF: $T_0 = 0, \quad T_n = 2T_{n-1} + 1$

Divide RF by 2^n

$\frac{T_0}{2^0} = 0, \quad \frac{T_n}{2^n} = \frac{2T_{n-1}}{2^n} + \frac{1}{2^n}$

and we get

$\frac{T_0}{2^0} = 0, \quad \frac{T_n}{2^n} = \frac{T_{n-1}}{2^{n-1}} + \frac{1}{2^n}$

Denote $S_n = \frac{T_n}{2^n}$, we get a recursive sum formula SR

RS: $S_0 = 0, \quad S_n = S_{n-1} + \frac{1}{2^n}$

From RF to Sum S_n to CF

SR: $S_0 = 0, \quad S_n = S_{n-1} + \frac{1}{2^n}$

It means that $S : N \rightarrow R$ and

$$S_n = \sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n} \qquad \text{(as } S_n \text{ is geometric)}$$

But we have $S_n = \frac{T_n}{2^n}$ so we get

$$\boxed{T_n = 2^n S_n}$$

and we evaluate

$$\boxed{T_n = 2^n - 1} \leftarrow \text{CF for RF}$$

Tower of Hanoi Revisited

RF: $T_0 = 0, \quad T_n = 2T_{n-1} + 1$

We have proved in **Chapter 1** that

$$\boxed{T_n = 2^n - 1} \leftarrow \text{Closed Formula}$$

We now **reverse** the the previous problem:

we will get a sum S_n and its **closed formula** from the closed formula CF for T_n

Divide T_n formula by 2^n

$\frac{T_0}{2^0} = 0, \quad \frac{T_n}{2^n} = \frac{2T_{n-1}}{2^n} + \frac{1}{2^n}$

Put $S_n = \frac{T_n}{2^n}$ and we get

$$\text{SR:} \quad S_0 = 0, \quad S_n = S_{n-1} + \frac{1}{2^n}$$

Now, $S_n = \frac{T_n}{2^n}$ and using CF for T_n we get $S_n = \frac{2^n - 1}{2^n}$

Thus, $\boxed{S_n = \sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n}} \leftarrow \text{SUM}$