

cse547, math547
DISCRETE MATHEMATICS

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LECTURE 4a

CHAPTER 1, Problem 20
SOLUTION

Problem

Use the **repertoire method** to solve the general **five-parameter recurrence RF**

Solve means FIND the closed formula **CF** equivalent to following **RF**

$$h(1) = \alpha;$$

$$h(2n + 0) = 4h(n) + \gamma_0 n + \beta_0;$$

$$h(2n + 1) = 4h(n) + \gamma_1 n + \beta_1, \text{ for all } n \geq 1.$$

General Form of CF

Our RF for h is a FIVE parameters function and it is a **generalization** of the General Josephus GJ function f considered before

So we **guess** that now the **general form** of the CF is also a generalization of the one we already proved for GJ , i.e.

General form of CF is

$$h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$$

The **Problem 20** asks us to use the **repertoire method** to prove that CF is **equivalent** to RF

Thinking Time

Solution requires a system of **10 equations** on $\alpha, \gamma_0, \beta_0, \gamma_1, \beta_1, A(n), B(n), C(n), D(n), E(n)$ and accordingly a **5 repertoire functions**

Let's **THINK a bit** before we embark on quite complicated calculations and without certainty that they would succeed (look at the solution to the **Problem 16** in Lecture 4)

First : we observe that when when $\gamma_0 = \gamma_1 = 0$, we get that the function h becomes for Generalize Josephus function f below for $k = 4$:

$$f(1) = \alpha, \quad f(2n + j) = kf(n) + \beta_j,$$

where $k \geq 2, j = 0, 1$ and $n \geq 0$

It seems **worth to examine first** the case $\gamma_0 = \gamma_1 = 0$

GJ f Closed Formula Solution

We **proved** that GJ function f has a **relaxed k-representation** closed formula

$$f((\mathbf{1}, \mathbf{b}_{m-1}, \dots, \mathbf{b}_1, \mathbf{b}_0)_2) = (\alpha, \beta_{\mathbf{b}_{m-1}}, \dots, \beta_{\mathbf{b}_0})_{\mathbf{k}}$$

where β_{b_j} are defined by

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} ; \quad j = 0, \dots, m-1,$$

for the **relaxed k-radix representation** defined as

$$(\alpha, \beta_{\mathbf{b}_{m-1}}, \dots, \beta_{\mathbf{b}_0})_{\mathbf{k}} = \alpha \mathbf{k}^m + \mathbf{k}^{m-1} \beta_{\mathbf{b}_{m-1}} + \dots + \beta_{\mathbf{b}_0}$$

Special Case of h

Consider now a **special case** of our **h**, when $\gamma_0 = \gamma_1 = 0$

We know that it now has a **relaxed 4 - representation** closed formula

$$h((1, b_{m-1}, \dots, b_1, b_0)_2) = (\alpha, \beta_{b_{m-1}}, \dots, \beta_{b_0})_4$$

It means that we get

Fact 0 For any $n = (1, b_{m-1}, \dots, b_1, b_0)_2$,

$$h(n) = (\alpha, \beta_{b_{m-1}}, \dots, \beta_{b_0})_4$$

Observe that our general form of **CF** in this case becomes

$$h(n) = \alpha A(n) + \beta_0 D(n) + \beta_1 E(n)$$

We must have $h(n) = h(n)$, for all n , so from this and **Fact 0** we get the following equation 1 (stated as Fact 1)

Equation 1

Fact 1 For any $n = (1, b_{m-1}, \dots, b_1, b_0)_2$,

$$\alpha A(n) + \beta_0 D(n) + \beta_1 E(n) = (\alpha, \beta_{b_{m-1}}, \dots, \beta_{b_0})_4$$

This provides us with the **Equation 1** for finding our general form of **CF**

Next Observation

Observe that $A(n)$ in the Original Josephus was proved to be given by a formula

$$A(n) = 2^k, \text{ for all } n = 2^k + \ell, \ 0 \leq \ell < 2^k$$

So we wonder if we could have a **similar solution** for our $A(n)$

Special Case of h

We evaluate now few initial values for h in case $\gamma_0 = \gamma_1 = 0$

$$h(1) = \alpha;$$

$$\begin{aligned} h(2) &= h(2(1) + 0) = 4h(1) + \beta_0 \\ &= 4\alpha + \beta_0; \end{aligned}$$

$$\begin{aligned} h(3) &= h(2(1) + 1) = 4h(1) + \beta_1 \\ &= 4\alpha + \beta_1; \end{aligned}$$

$$\begin{aligned} h(4) &= h(2(2) + 0) = 4h(2) + \beta_0 \\ &= 16\alpha + 5\beta_0; \end{aligned}$$

Equation 2

It is pretty obvious that we do have a similar formula for $A(n)$ as on the Original Josephus **OJ**

We write it as the next

Fact 2

For all $n = 2^k + \ell$, $0 \leq \ell < 2^k$, $n \in N - \{0\}$

$$A(n) = 4^k$$

This provides us with the **Equation 2** for finding our general form of **CF**

Repertoire Method

The **proof** of **Fact 2** is almost identical to the one in the case of **OJ**, and for the **Problem** in Lecture 4, so leave it as an exercise

We have already developed **2 Equations** (as stated in **Facts 1, 2**) so we need now to **consider** only **3 repertoire functions** to obtain **all Equations** need to solve the problem

Repertoire Function 1

We return now to our **original functions**:

$$\text{RF: } h(1) = \alpha, h(2n) = 4h(n) + \gamma_0 n + \beta_0,$$

$$h(2n + 1) = 4h(n) + \gamma_1 n + \beta_1,$$

$$\text{CF: } h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$$

Consider a **first repertoire function** : $\mathbf{h(n) = 1}$, for all $n \in N - \{0\}$

We put $\mathbf{h(n) = h(n) = 1}$, for all $n \in N - \{0\}$

We have $\mathbf{h(1) = 1}$, and $h(1) = \alpha$, so we get $\alpha = 1$

We now use $\mathbf{h(n) = h(n) = 1}$, for all $n \in N - \{0\}$ and evaluate

$$h(2n) = 4h(n) + \gamma_0 n + \beta_0$$

$$1 = 4 + \gamma_0 n + \beta_0$$

$$0 = (3 + \beta_0) + \gamma_0 n$$

$$h(2n + 1) = 4h(n) + \gamma_1 n + \beta_1;$$

$$1 = 4 + \gamma_1 n + \beta_1$$

$$0 = (3 + \beta_1) + \gamma_1 n$$

We get $\gamma_0 = \gamma_1 = 0$, $\beta_0 = \beta_1 = -3$

Solution 1: $\alpha = 1$, $\gamma_0 = \gamma_1 = 0$, $\beta_0 = \beta_1 = -3$

Equation 3

The general form of **CF** is:

$$h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$$

We put $h(n) = \mathbf{h(n)} = \mathbf{1}$, for all $n \in N - \{0\}$, i.e.

$\alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n) = \mathbf{h(n)} = \mathbf{1}$, for all $n \in N - \{0\}$, where $\alpha, \gamma_1, \beta_0, \gamma_2, \beta_1$ already are evaluated in the **Solution 1** as $\alpha = 1, \gamma_0 = \gamma_1 = 0, \beta_0 = \beta_1 = -3$

We get

CF = **RF** if and only if the following holds

Fact 3 For all $n \in N - \{0\}$,

$$A(n) - 3D(n) - 3E(n) = 1$$

This is our **Equation 3**

Repertoire Function 2

Consider a **repertoire function 2**: $h(n) = n$, for all $n \in N - \{0\}$

We put $h(n) = h(n) = n$, for all $n \in N - \{0\}$

$h(1) = \alpha$, $h(1) = 1$ and $h(n) = h(n)$, hence $\alpha = 1$

We now use $h(n) = h(n) = n$, for all $n \in N - \{0\}$ and evaluate

$$h(2n) = 4h(n) + \gamma_0 n + \beta_0$$

$$2n = 4n + \gamma_0 n + \beta_0$$

$$0 = (\gamma_0 + 2)n + \beta_0$$

$$h(2n + 1) = 4h(n) + \gamma_1 n + \beta_1;$$

$$2n + 1 = 4n + \gamma_1 n + \beta_1$$

$$0 = (\gamma_1 + 2)n + (\beta_1 - 1)$$

We get $\gamma_0 = \gamma_1 = -2$, $\beta_0 = 0$, $\beta_1 = 1$ and

Solution 2: $\alpha = 1$, $\gamma_0 = \gamma_1 = -2$, $\beta_0 = 0$, $\beta_1 = 1$

Equation 4

$$\text{CF: } h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$$

We evaluate **CF** for $h(n) = \mathbf{h(n) = n}$, for all $n \in N - \{0\}$ and for the **Solution 2**: $\alpha = 1, \gamma_0 = \gamma_1 = -2, \beta_0 = 0, \beta_1 = 1$ and get

CF = **RF** if and only if the following holds

Fact 4 For all $n \in N - \{0\}$

$$A(n) - 2B(n) - 2C(n) + E(n) = n$$

This is our **Equation 4**

Repertoire Function 3

Consider a **repertoire function 3**: $h(n) = n^2$, for all $n \in N$

We put $h(n) = h(n) = n^2$, for all $n \in N - \{0\}$

$h(1) = \alpha$, $h(1) = 1$, hence $\alpha = 1$

$$h(2n + 0) = 4h(n) + \gamma_0 n + \beta_0$$

$$(2n)^2 = 4n^2 + \gamma_0 n + \beta_0$$

$$4n^2 = 4n^2 + \gamma_0 n + \beta_0$$

$$0 = \gamma_0 n + \beta_0$$

$$h(2n + 1) = 4h(n) + \gamma_1 n + \beta_1;$$

$$(2n + 1)^2 = 4n^2 + \gamma_1 n + \beta_1$$

$$4n^2 + 4n + 1 = 4n^2 + \gamma_1 n + \beta_1$$

$$0 = (\gamma_1 - 4)n + (\beta_1 - 1)$$

We get $\gamma_0 = 0$, $\gamma_1 = 4$, $\beta_0 = 0$, $\beta_1 = 1$ and

Solution 3: $\alpha = 1$, $\gamma_0 = 0$, $\gamma_1 = 4$, $\beta_0 = 0$, $\beta_1 = 1$

Equation 5

$$\text{CF: } h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$$

We evaluate CF for $h(n) = \mathbf{h(n)} = \mathbf{n^2}$, for all $n \in N - \{0\}$

and for the **Solution 3**:

$$\alpha = 1, \gamma_0 = 0, \gamma_1 = 4, \beta_0 = 0, \beta_1 = 1$$

We get **CF = RF** if and only if the following holds

Fact 5 For all $n \in N - \{0\}$

$$A(n) + 4C(n) + E(n) = n^2$$

This is our **Equation 5**

Repertoire Method: System of Equations

We obtained the following system of **5 equations** on $A(n)$, $B(n)$, $C(n)$, $D(n)$, $E(n)$

1. $\alpha A(n) + \beta_0 D(n) + \beta_1 E(n) = (\alpha, \beta_{b_{m-1}}, \dots, \beta_{b_0})_4$

2. $A(n) = 4^k$

3. $A(n) - 3D(n) - 3E(n) = 1$

4. $A(n) - 2B(n) - 2C(n) + E(n) = n$

5. $A(n) + 4C(n) + E(n) = n^2$

We solve it and put the solution into

$$h(n) = \alpha A(n) + \gamma_0 B(n) + \gamma_1 C(n) + \beta_0 D(n) + \beta_1 E(n)$$

System of Equations Solution

First solution formulas

$$A(n) = 4^k$$

$$B(n) = \frac{A(n) - 3C(n) - 1}{3} \quad (\text{from } \mathbf{3.})$$

$$D(n) = \frac{3A(n) + 3C(n) - n^2 - 2n}{4}$$

$$E(n) = \frac{n^2 - A(n) - C(n)}{4} \quad (\text{from } \mathbf{5.})$$

$$C(n) = ((\alpha\beta_{b_{m-1}} \dots \beta_{b_1} \beta_{b_0})_4 - \alpha \cdot 4^k - \beta_0(1 - 4^k)/3)/(\beta_0 + \beta_1)$$

Problem 20 Solution

After substitution we obtain:

$$A(n) = 4^k$$

$$B(n) = \frac{4^k - 3C(n) - 1}{3}$$

$$D(n) = \frac{3 \cdot 4^k + 3C(n) - n^2 - 2n}{4}$$

$$E(n) = \frac{n^2 - 4^k - C(n)}{4}$$

$$C(n) = ((\alpha\beta_{b_{m-1}} \dots \beta_{b_1}\beta_{b_0})_4 - \alpha \cdot 4^k - \beta_0(1 - 4^k)/3)/(\beta_0 + \beta_1)$$

General Remark

Observe that the use of the **Relaxed k -Radix Representation** partial solution, i.e. the **equation 1** as well as proving the formula for $A(n)$, i.e. the **equation 2** were **essential** for the **solution of the problem**

Like in Ch1 **Problem 16** - the other typical repertoire functions like $h(n) = n^3$ etc . **FAIL**, i.e. lead to contradictions - **easy and left for you to evaluate!**

Read carefully published Solutions to Ch1**Problem 16**, which is, in fact an easier case of **Problem 20** just presented in a **FULL DETAIL**

Observe also that we have **proved** all **Relaxed k -Radix Representation** formulas needed and you have to **KNOW** **these proofs for your tests**