cse547,
DISCRETE MATHEMATICS

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LECTURE 4

CHAPTER 1

PART FIVE: Binary and Relaxed Binary Solutions for Generalized Josephus

Binary Solution

We proved that the **original** J-recurrence:

J(1) = 1, J(2n) = 2J(n) - 1, J(2n+1) = 2J(n) + 1 for $n > 1$

has a beautiful **binary** CF **solution**

$$J((b_m, b_{m-1}, ...b_1, b_0)_2) = (b_{m-1}, b_{m-2}, ...b_0, b_m)_2,$$

move b_m !

where $b_m = 1$, as $n = 2^m + l$

Question: Does the **generalized Josephus** GJ admits a similar solution?

Answer: YES.

Generalized Josephus GF

We **generalized** the function J to function
$f: N - \{0\} \longrightarrow N$ defined as follows

$$f(1) = \alpha$$

$$f(2n) = 2f(n) + \beta, \quad n \geq 1$$

$$f(2n+1) = 2f(n) + \gamma, \quad n \geq 1$$

Observe that J = f for $\alpha = 1, \beta = -1, \gamma = 1$

We call the function f a **Generalized Josephus** GJ

New Formula for GJ

We re-write the function f as follows

$$f(1) = \alpha;$$
$$f(2n + j) = 2f(n) + \beta_j$$
$$\text{for } \ j = 0, 1, \qquad\qquad n \geq 1$$

Assume

$$k = (b_m, b_{m-1}, ...b_1, b_0)_2$$

We want to evaluate:

$$f(k) = f((b_m, b_{m-1}, ...b_1, b_0)_2)$$

Binary Representation for k=2n

Consider case when

$$k = 2n + 0, \quad j = 0.$$

The binary representation of $k = 2n$ is given as:

$$2n = (b_m, b_{m-1}, ...b_1, b_0)_2$$

$$2n = 2^m b_m + 2^{m-1} b_{m-1} + ... + 2b_1 + b_0$$

Binary Representation for k=2n

We get $b_m = 1$ and $b_0 = 0$

Hence,

$$n = 2^{m-1} b_m + ... + b_1$$

$$\mathbf{n = (b_m, b_{m-1}, ...b_1)_2}$$

Question: What happens when $k = 2n + 1, \ \ j = 1$?

Binary Representation for k=2n+1

Consider case when $k = 2n + j,\ j = 1$

The binary representation of k=2n +1 is given as:

$$2n + 1 = (b_m, b_{m-1}, ...b_1, b_0)_2$$

$$2n + 1 = 2^m b_m + 2^{m-1} b_{m-1} + ... + 2b_1 + b_0$$

$$b_0 = 1, b_m = 1$$

Binary Representation for k=2n+1

We get

$$2n + 1 = 2^m b_m + 2^{m-1} b_{m-1} + ... + 2b_1 + 1$$

$$2n = 2^m b_m + 2^{m-1} b_{m-1} + ... + 2b_1$$

$$n = 2^{m-1} b_m + 2^{m-1} b_{m-1} + ... + b_1$$

$$\mathbf{n = (b_m, b_{m-1}, ...b_1)_2}$$

Binary Representation

We have **proved** that whether we have a binary representation of $2n = (b_m, b_{m-1}, ...b_1, b_0)_2$

or a binary representation of $2n+1 = (b_m, b_{m-1}, ...b_1, b_0)_2$,

the corresponding representations of n are the same:

$$\mathbf{n} = (\mathbf{b_m}, \mathbf{b_{m-1}}, ...\mathbf{b_1})_2$$

Fact

When dealing with **binary representation** we do not need to consider cases of $\mathbf{n} \in$ **odd** or $\mathbf{n} \in$ **even**

when using our recursive formula

$$f(2n + j) = 2f(n) + \beta_j, \qquad j = 0, 1$$

CF in Binary Representation

Here is our recursive formula

RF: $f(1) = \alpha, \quad f(2n) = 2f(n) + \beta_0, \quad f(2n+1) = 2f(n) + \beta_1$

By the **Fact** evaluate can write RF using n in **binary representation**

$$f((b_m, b_{m-1}, ...b_1, b_0)_2) = 2f((b_m, b_{m-1}, ...b_1)_2) + \beta_{b_i},$$

where

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} \qquad j = 0...m-1$$

CF in Binary Representation

We evaluate:

$$\begin{aligned}
\mathbf{f((b_m, b_{m-1}, ...b_1, b_0)_2)} &= 2f((b_m, b_{m-1}, ..., b_1)_2) + \beta_{b_0} \\
&= 2(2f((b_m, b_{m-1}, ..., b_2)_2) + \beta_{b_1}) + \beta_{b_0} \\
&= 4f((b_m, b_{m-1}, ..., b_2)_2) + 2\beta_{b_1} + \beta_{b_0} \\
&\vdots \\
&= 2^m f((b_m)_2) + 2^{m-1}\beta_{b_{m-1}} + ... + 2\beta_{b_1} + \beta_{b_0} \\
&= \mathbf{2^m f((1)_2) + 2^{m-1}\beta_{b_{m-1}} + ... + 2\beta_{b_1} + \beta_{b_0}}
\end{aligned}$$

CF in Binary Representation

We know that $f(1) = \alpha$

So we get (almost) CF formula

$$\mathbf{f((b_m, b_{m-1}, ...b_1, b_0)_2) = 2^m}\alpha + \mathbf{2^{m-1}}\beta_{\mathbf{b_{m-1}}} + ... + \mathbf{2}\beta_{\mathbf{b_1}} + \beta_{\mathbf{b_0}}$$

where

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} \qquad j = 0...m-1$$

Relaxed Binary CF

We **define** a **relaxed binary** representation as follows

$$\mathbf{2^m}\alpha + \mathbf{2^{m-1}}\beta_{\mathbf{b_{m-1}}} + ... + \beta_{\mathbf{b_0}} = (\alpha, \beta_{\mathbf{b_{m-1}}}, ..\beta_{\mathbf{b_0}})_\mathbf{2}$$

where now β_{b_k} are now any numbers, not only 0,1

We write the **relaxed binary** CF as

$$\begin{aligned} \mathbf{f((b_m, b_{m-1}, ...b_1, b_0)_2)} &= (\alpha, \beta_{\mathbf{b_{m-1}}}, ..\beta_{\mathbf{b_0}})_\mathbf{2} \\ \textit{"normal"} &= \textit{relaxed} \end{aligned}$$

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} \qquad j = 0, ..., m-1$$

Example: Original Josephus

The GJ function f becomes the **original Josephus** when $\beta_0 = -1, \beta_1 = 1$

Example

Let n = 100

Use the **relaxed binary** CF to show that f(100) = 73 =J(n) as we have already evaluated

$$n = (\,1\;1\;0\;0\;1\;0\;0\,)_2$$
$$(b_6 b_5 b_4 b_3 b_2 b_1 b_0)$$

Relaxed coordinates are

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases}$$ and hence

$$\beta_{b_j} = \begin{cases} -1 & b_j = 0 \\ 1 & b_j = 1 \end{cases}$$

Example

We have

$$\begin{aligned} n &= (\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ )_2 \\ & \quad (b_6 b_5 b_4 b_3 b_2 b_1 b_0) \end{aligned}$$

$$\begin{aligned} \mathbf{f}((\mathbf{b_m}, \mathbf{b_{m-1}}, ...\mathbf{b_1}, \mathbf{b_0})_\mathbf{2}) &= (\alpha, \beta_{\mathbf{b_{m-1}}}, ...\beta_{\mathbf{b_0}})_\mathbf{2} \\ \text{"}\textit{normal}\text{"} &= \textit{relaxed} \end{aligned}$$

$$\beta_{b_j} = \begin{cases} -1 & b_j = 0 \\ 1 & b_j = 1 \end{cases}$$

We evaluate

$$f(n) = f((\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ )_2)) =^{relax} (\alpha,\ \beta_{b_5},\ \ldots\ \beta_{b_0})$$

$$= (1, 1, -1, -1, 1, -1, -1)_2 = 64 + 32 - 16 - 8 + 4 - 2 - 1 = 73$$

Cyclic - Shift Property

We **proved** that the original J-recurrence:

J(1) = 1, J(2n) = 2J(n) - 1, J(2n+1) = 2J(n) + 1 for $n > 1$

has a beautiful binary CF solution, called **cyclic - shift property**, namely

$$J((b_m, b_{m-1}, ...b_1, b_0)_2) = (b_{m-1}, b_{m-2}, ...b_0, b_m)_2$$

We prove now that the **cyclic - shift property** holds also for the GF function f in the case when $\beta_0 = -1, \beta_1 = 1$, i.e.

$$f((b_m, b_{m-1}, ...b_1, b_0)_2) = (b_{m-1}, b_{m-2}, ...b_0, b_m)_2$$

We know that $b_m = 1$, so we have to prove that:

$$f(1, b_{m-1}, ...b_1, b_0)_2) = (b_{m-1}, b_{m-2}, ...b_0, 1)_2,$$

for f such that $\beta_0 = -1, \beta_1 = 1$

Cyclic - Shift Property for GJ

We have proved the **relaxed binary** CF solution for GJ:

$$CF: \quad f((1, b_{m-1}, ...b_1, b_0)_2) = (1, \beta_{b_{m-1}}, ...\beta_{b_0})_2$$

where $f(n)$ contains now 1 and -1 as defined by

$$\beta_{b_j} = \begin{cases} -1 & b_j = 0 \\ 1 & b_j = 1 \end{cases}$$

Example

EXAMPLE

Consider $n = (1, 0, 0, 1, 0, 0, 1)_2$

By CF we have that

$$f((1, 0, 0, 1, 0, 0, 1)_2) = (1, -1, -1, 1, -1, -1, 1)_2$$

General Observation

f transforms a BLOCK of 0's in **normal binary representation** into a BLOCK of -1's in the **relaxed representation**

$$\mathbf{f((1, 0, 0 ... 0)_2) = (1, -1, -1 ... -1)_2}$$

ONE BLOCK Transformation

We **prove** now the following relationship between relaxed and **normal** representation

ONE BLOCK **transformation**

$$(\mathbf{1}, \mathbf{-1}, \mathbf{-1} ..., \mathbf{-1})_{\mathbf{2}} = (\mathbf{0}, \mathbf{0}, \mathbf{0} ..., \mathbf{0}, \mathbf{1})_{\mathbf{2}}$$

Proof: Let $n = ((-1, -1 ..., -1)_2$

$$\begin{aligned}
\mathbf{n} = (\mathbf{1}, \mathbf{-1}, \mathbf{-1} ..., \mathbf{-1})_{\mathbf{2}} &=^{def} 2^m - 2^{m-1} - 2^{m-2} - ... - 2^1 - 2^0 \\
&= 2^{m-1} - 2^{m-2} - ... - 2^1 - 2^0 \\
&= 2^{m-2} - 2^{m-3} - ... - 2^1 - 2^0 \\
&\vdots \\
&= 2^1 - 2^0 \\
&= 2 - 1 \\
&= \mathbf{1} = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1})_{\mathbf{2}}
\end{aligned}$$

Many Blocks Transformation

Example for TWO BLOCKS **transformation** plus **binary shift**

$$
\begin{aligned}
f((1,0,0,1,1,0,0,1)_2) &= (1,-1,-1,1,1,-1,-1,1)_2 \\
&=^{1bt} (0,0,1,1,1,-1,-1,1)_2 \\
&=^{1bt} (0,0,1,1,0,0,1,1)_2 \\
&= (0,0,1,1,0,0,1,1)_2
\end{aligned}
$$

We know that $f((b_m,...,b_1,b_0)_2) = (\alpha, \beta_{b_{m-1}},...,\beta_{b_0})_2$ **OBSERVE** that each block of binary digits $(1,0..0)_2$ is **transformed** by f into $(1,-1,...)_2$ and multiple applications of **one block transformation** transforms them **back** to $(1,0..0)_2$, so

$$((\alpha, \beta_{\mathbf{b_{m-1}}}, \ldots, \beta_{\mathbf{b_0}})_{\mathbf{2}} =^{\mathbf{mbt}} (\mathbf{b_{m-1}}, ...\mathbf{b_1}, \mathbf{b_0}, \mathbf{1})_{\mathbf{2}}$$

where mbt denotes multiple BLOK transformations, and we know that $\alpha = 1$

Cyclic - Shift Property

We now evaluate:

$$\begin{aligned} f((1, b_{m-1}, ..., b_1, b_0)_2) &= (\alpha, \beta_{b_{m-1}}, ..., \beta_{b_0})_2 \\ &=^{mbt} (b_{m-1}, ..., b_1, b_0, 1)_2 \end{aligned}$$

This ends the proof of the Cyclic - Shift Property for Generalized Josephus f with $\alpha = 1,\ \beta_0 = -1,\ \beta_1 = 1$

Exercise 1

Given

$$\begin{aligned} f(1) &= 5 \\ f(2n) &= 2f(n) - 10 \\ f(2n+1) &= 2f(n) + 83 \end{aligned}$$

Exercise 1

Evaluate f(100)

Solution: just apply proper formulas!

Exercise 2

Given

$$\begin{aligned} f(1) &= 5 \\ f(2n) &= 3f(n) - 10 \\ f(2n+1) &= 3f(n) + 83 \end{aligned}$$

Exercise 2

Evaluate f(100)

Observe that now we don't have proper formulas! They work only for base 2!

Goal **Generalize** f and develop **new formulas** (if possible)

RADIX Representation

We **proved** while solving the Generalized Josephus that

RF: $f(1) = \alpha, \quad f(2n + j) = 2f(n) + \beta_j$

where $j = 0, 1$ and $n \geq 0$

has a **relaxed binary** CF formula

$$CF: \quad \mathbf{f((1, b_{m-1}, ...b_1, b_0)_2)} = (\alpha, \beta_{\mathbf{b_{m-1}}}, ...\beta_{\mathbf{b_0}})_{\mathbf{2}}$$

where β_{b_j} are defined by

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} \qquad j = 0, ..., m-1$$

and where the **relaxed binary representation** is defined as

$$(\alpha, \beta_{\mathbf{b_{m-1}}}, ..., \beta_{\mathbf{b_0}})_{\mathbf{2}} = \mathbf{2^m}\alpha + \mathbf{2^{m-1}}\beta_{\mathbf{m-1}} + ... + \beta_{\mathbf{b_0}}$$

Relaxed Radix Representation

We **generalize** GJ as follows

RF: $f(1) = \alpha, \quad f(2n+j) = kf(n) + \beta_j,$

where $k \geq 2, \quad j = 0, 1$ and $n \geq 0$

Exercise: PROVE that RF has a **relaxed** k-**representation** closed formula

$$CF: \quad \mathbf{f((1, b_{m-1}, ...b_1, b_0)_2)} = (\alpha, \beta_{\mathbf{b_{m-1}}}, ...\beta_{\mathbf{b_0}})_{\mathbf{k}}$$

where β_{b_j} are defined as before by

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} ; \quad j = 0, ..., m-1$$

and where we define the **relaxed** k- **representation** as follows

Relaxed k-Radix Representation

Definition

A relaxed k- **representation** is defined as

$$(\alpha, \beta_{b_{m-1}}, ..., \beta_{b_0})_k = \alpha k^m + k^{m-1}\beta_{m-1} + ... + \beta_{b_0}$$

We repeat the **proof i** directly from there definition following the proof for the case $k = 2$

Proof

$$
\begin{aligned}
f((b_m, b_{m-1}, ..., b_1, b_0)_2) &= k f((b_m, b_{m-1}, ..., b_1)_2) + \beta_{b_0} \\
&= k(k f((b_m, b_{m-1}, ..., b_2)_2) + \beta_{b_1}) + \beta_{b_0} \\
&= k^2 f((b_m, b_{m-1}, ..., b_2)_2) + k\beta_{b_1} + \beta_{b_0} \\
&= k^3 f((b_m, b_{m-1}, ..., b_3)_2) + k^2\beta_{b_2} + k\beta_{b_1} + \beta_{b_0} \\
&\quad \vdots \\
&= k^m f((b_m)_2) + k^{m-1}\beta_{b_{m-1}} + ... + k\beta_{b_1} + \beta_{b_0} \\
&= k^m \alpha + k^{m-1}\beta_{b_{m-1}} + ... + k^2\beta_{b_2} + k\beta_{b_1} + \beta_{b_0} \\
&= (\alpha, \beta_{b_{m-1}}, ..., \beta_{b_1}, \beta_{b_0})_k
\end{aligned}
$$

$$
\begin{aligned}
f((b_m, b_{m-1}, ..., b_1, b_0)_2) &= (\alpha, \beta_{b_{m-1}} ... \beta_{b_1}, \beta_{b_0})_k \\
\text{base 2} &\rightarrow \text{base k}
\end{aligned}
$$

Example

Given RF:

$$\begin{aligned} f(1) &= 5 \\ f(2n) &= 6f(n)+3 \\ f(2n+1) &= 6f(n)-10 \end{aligned}$$

Evaluate: $f(100)$ by the use of the k- **representation** and closed formula

$$CF: \quad \mathbf{f((1, b_{m-1}, ...b_1, b_0)_2) = (\alpha, \beta_{b_{m-1}}, ...\beta_{b_0})_k}$$

where β_{b_j} are defined as before by

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} \quad ; \quad j = 0, ..., m-1$$

Example Solution

Given

$$\begin{aligned} f(1) &= 5 \\ f(2n) &= 6f(n) + 3 \\ f(2n+1) &= 6f(n) - 10 \end{aligned}$$

We evaluate

$$\alpha = 5$$

$$\beta_0 = 3$$

$$\beta_1 = -10$$

Example Solution

Evaluate: $f(100)$

$$\alpha = 5,\ \beta_0 = 3,\ \beta_1 = -10,\ k = 6,\ n \quad = \quad (\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ )_2$$
$$(b_6 b_5 b_4 b_3 b_2 b_1 b_0)$$

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases}, \qquad j = 0, ..., m-1,$$

$$\beta_{b_0} = 3,\ \ \beta_{b_1} = 3,\ \ \beta_{b_2} = -10,\ \ \beta_{b_3} = 3;\ \ \beta_{b_4} = 3,$$
$$\beta_{b_5} = -10,\ \ \alpha = 5$$

$$\mathbf{f(100) = f((\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ )_2) = (5, -10, 3, 3, -10, 3, 3)_6}$$

More General GJ Function

Further Generalization of GJ

RF:

$$f(i) = \alpha_i, \qquad i = 1, ..., d-1$$
$$f(dn + j) = cf(n) + \beta_j, \qquad n \geq 1, \;\; 0 \leq j < d$$

Exercise

Prove the following closed formula

CF:

$$\mathbf{f((b_m, b_{m-1}, ..., b_1, b_0)_d) \quad = \quad (\alpha_{b_m}, \beta_{b_{m-1}} ... \beta_{b_1}, \beta_{b_0})_c}$$

Example

$$
\begin{aligned}
f(1) &= 34 \\
f(2) &= 5 \\
f(3n) &= 10f(n) + 76 \\
f(3n+1) &= 10f(n) - 2 \\
f(3n+2) &= 10f(n) + 8
\end{aligned}
$$

$$
\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \\ \beta_2 & b_j = 2 \end{cases}, \qquad j = 0, ..., d-1,
$$

Example Solution

We evaluate:

$$i = 1, 2$$
$$j = 0, 1, 2$$

$$d = 3$$
$$c = 10$$

$$\alpha_1 = 34$$
$$\alpha_2 = 5$$
$$\beta_0 = 76$$
$$\beta_1 = -2$$
$$\beta_2 = 8$$

Example

Evaluate: $f(19)$

$$19 = (201)_3 = 2 \cdot 3^2 + 0 \cdot 3 + 1$$

$$\alpha_{b_2} = \alpha_2 = 5$$
$$\beta_{b_0} = \beta_0 = 76$$
$$\beta_{b_1} = \beta_1 = -2$$

$$\begin{aligned} f(19) &= f((201)_3) \\ &= (5, 76, -2)_{10} \\ &= 5 \cdot 10^2 + 76 \cdot 10 - 2 \\ &= 500 + 760 - 2 \\ &= 1258 \end{aligned}$$

Short Solution

$$\mathbf{f}((\mathbf{b_m}, \mathbf{b_{m-1}}, ..., \mathbf{b_1}, \mathbf{b_0})_\mathbf{d}) \quad = \quad (\alpha_{\mathbf{b_m}}, \beta_{\mathbf{b_{m-1}}} .. \beta_{\mathbf{b_1}}, \beta_{\mathbf{b_0}})_\mathbf{c}$$

Take

$$19 = (2\ 0\ 1)_3$$

Corresponding solution is

$$(\alpha_2,\ \ \beta_0,\ \ \beta_1)_{10}$$

we evaluate $\alpha_2 = 5,\ \ \beta_0 = 76,\ \ \beta_1 = -2$ and get

Solution:

$$(5,\ \ 76,\ \ -2)_{10}$$

New Generalization of GJ

Problem

Use the repertoire method to **solve** the following yet more general **four-parameter recurrence** RF

$$
\begin{aligned}
h(1) &= \alpha; \\
h(2n+0) &= 3h(n) + \gamma n + \beta_0; \\
h(2n+1) &= 3h(n) + \gamma n + \beta_1, \quad \text{for all } \ n \geq 1.
\end{aligned}
$$

Solve means FIND a **closed formula** CF equivalent to RF

General Form of CF

Our RF is a FOUR parameters function and it is a **generalization** of the General Josephus GJ function f considered before

So we guess that now the **general form** of the CF is also a generalization of the one we already proved for GJ , i.e.

General form of CF is

$$h(n) = \alpha A(n) + \gamma B(n) + \beta_0 C(n) + \beta_1 D(n)$$

The **Problem** asks us to use repertoire method to prove that CF is **equivalent** to RF

Thinking Time

Solution requires us to develop a system of **8 equations** on α, γ, β_0, β_1, A(n), B(n), C(n), D(n)

and accordingly a **4 repertoire functions**!

First : we observe that when $\gamma = 0$, we get that h becomes for Generalized Josephus function f below for k = 3:

$f(1) = \alpha, \quad f(2n + j) = kf(n) + \beta_j,$

where $k \geq 2$, $j = 0, 1$ and $n \geq 0$

It seems **worth to examine** the case $\gamma = 0$ **first**

Closed Formula for GJ function f

We **proved** that GJ function f has the **relaxed k- representation** closed formula

$$\mathbf{f((1, b_{m-1}, ...b_1, b_0)_2)} = (\alpha, \beta_{\mathbf{b_{m-1}}}, ...\beta_{\mathbf{b_0}})_{\mathbf{k}}$$

where β_{b_j} are defined by

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} \quad ; \qquad j = 0, ..., m-1,$$

for the **relaxed k- radix representation** defined as

$$(\alpha, \beta_{\mathbf{b_{m-1}}}, ..., \beta_{\mathbf{b_0}})_{\mathbf{k}} = \alpha \mathbf{k^m} + \mathbf{k^{m-1}} \beta_{\mathbf{m-1}} + ... + \beta_{\mathbf{b_0}}$$

Special Case of the function h

Consider now a special case of the function h for $\gamma = 0$

We know that it now has a relaxed 3 - representation closed formula

$$h((1, b_{m-1}, ...b_1, b_0)_2) = (\alpha, \beta_{b_{m-1}}, ...\beta_{b_0})_3$$

It means that we get

Fact 0 For any $n = (1, b_{m-1}, ...b_1, b_0)_2$,

$$h(n) = (\alpha, \beta_{b_{m-1}}, ...\beta_{b_0})_3$$

Observe that our general form of CF in this case becomes

$$h(n) = \alpha A(n) + \beta_0 C(n) + \beta_1 D(n)$$

We must have h(n) = h(n), for all n, inN so from this and **Fact 0** we get the following **Equation 1**

Equation 1

We must have h(n) = h(n), for all $n \in N$

From this and **Fact 0** we get the following

Fact 1 For any $n = (1, b_{m-1}, ...b_1, b_0)_2$,

$$\alpha A(n) + \beta_0 C(n) + \beta_1 D(n) = (\alpha, \beta_{b_{m-1}}, ...\beta_{b_0})_3$$

This provides us with the **Equation 1** for finding our general form of CF

Next Observation

Observe that A(n) in the Original Josephus was given (and proved to be) by a formula

$A(n) = 2^k$, for all $n = 2^k + \ell, \ \ 0 \leq \ell < 2^k$

We have a **similar solution** for our A(n)

Special Case of the function h

We evaluate now few initial values for h in case $\gamma = 0$

$$
\begin{aligned}
h(1) &= \alpha; \\
h(2) &= h(2(1)+0) = 3h(1)+\beta_0 \\
&= 3\alpha + \beta_0; \\[2ex]
h(3) &= h(2(1)+1) = 3h(1)+\beta_1 \\
&= 3\alpha + \beta_1; \\[2ex]
h(4) &= h(2(2)+0) = 3h(2)+\beta_0 \\
&= 9\alpha + 4\beta_0;
\end{aligned}
$$

Equation 2

It is pretty obvious that we do have a similar formula for A(n) as on the Original Josephus OJ

We write it as our **Fact 2** and get our

Fact 2

For all $n = 2^k + \ell, \;\; 0 \leq \ell < 2^k, \;\; n \in N - \{0\}$

$$A(n) = 3^k$$

The proof is almost identical to the one in the GJ, we re-write is here for our case as an exercise.

This provides us with the **Equation 2** for finding our general form of CF

Reminder

Reminder

We investigate the case when $\gamma = 0$, i.e. now our formulas are

RF: $h(1) = \alpha, \quad h(2n + j) = 3h(n) + \beta_j$

where $j = 0, 1$ and $n \geq 0$ and the closed formula is

CF: $h(n) = \alpha A(n) + \beta_0 C(n) + \beta_1 D(n)$

Proof of the Equation 2

Consider now a further case $\beta_0 = \beta_1$ =0, and $\alpha = 1$, i.e.

$RF: \quad h(1) = 1, \qquad h(2n) = 3h(n), \quad h(2n+1) = 3h(n)$

and $CF: \quad h(n) = A(n)$

We use $h(n) = A(n)$ and re-write RF in terms of A(n)

$RA: \quad A(1) = 1, \qquad A(2n) = 3A(n), \quad A(2n+1) = 3A(n)$

Fact Closed formula CAR for AR is:

CA: $A(n) = A(2^k + \ell) = 3^k, \quad 0 \leq \ell < 2^k$

Observe that this **Fact** is equivalent to the following **Fact 2**

Proof of the Fact 2

Fact 2 for all $n = 2^k + \ell, \; 0 \leq \ell < 2^k$

$$A(n) = 3^k$$

Proof by induction on k

Base case: k=0 i.e n=$2^0 + \ell, \; 0 \leq \ell < 1$, hence n = 1 and RA: A(1) = 1, and CA: A(1) = 3^0 = 1, so we have RA = CA

Inductive Assumption

$A(2^{k-1} + \ell)$ = $A(2^{k-1} + \ell) = 3^{k-1}$, for $0 \leq \ell < 2^{k-1}$

Inductive Thesis

$A(2^k + l)$ = $A(2^k + l) = 3^k$, for $0 \leq l < 2^k$

Two cases: $n \in even, \; n \in odd$

C1: $n \in even$

n := 2n, and we have $2^k + \ell = 2n$ iff $\ell \in even$

Proof of the Fact 2

We evaluate n as follows

$2n = 2^k + \ell, \quad n = 2^{k-1} + \frac{\ell}{2}$

We use n in the **inductive step**

Observe that the **correctness** of using $\frac{\ell}{2}$ follows from that fact that $\ell \in even$, so $\frac{\ell}{2} \in N$ and it can be proved formally like on the previous slides

Inductive Proof

$A(2n) =^{reprn} A(2^k + \ell) =^{n-eval} 3A(2^{k-1} + \frac{\ell}{2}) =^{ind}$
$3 * 3^{k-1} = 3^k$

Proof of the Fact 2

C2: $n \in odd$

n:= 2n+1, and we have $2^k + \ell = 2n + 1$ iff $\ell \in odd$

We evaluate n as follows

$2n + 1 = 2^k + \ell, \quad n = 2^{k-1} + \frac{\ell-1}{2}$

We use n in the **inductive step**

Observe that the correctness of using $\frac{\ell-1}{2}$ follows from that fact that $\ell \in odd$, so $\frac{\ell-1}{2} \in N$

Inductive Proof

$A(2n+1) =^{reprn} A(2^k + \ell) =^{n-eval} 3A(2^{k-1} + \frac{\ell-1}{2}) =^{ind}$
$3 * 3^{k-1} = 3^k$

It ends the proof of the **Fact 2:** $A(n) = 3^k$

Repertoire Method

We return now to **original functions**:

RF: $h(1) = \alpha, h(2n) = 3h(n) + \gamma n + \beta_0,$
$h(2n+1) = 3h(n) + \gamma n + \beta_1,$

CF: $h(n) = \alpha A(n) + \gamma B(n) + \beta_0 C(n) + \beta_1 D(n)$

We have already developed **two equations** (as stated in **Facts 1, 2**) so we need now to consider only **2 repertoire functions** to obtain 4 equations we need to solve the problem

Repertoire Function 1

Consider a **first repertoire function** : **h(n) = 1**, for all $n \in N - \{0\}$

We put h(n) = **h(n) = 1**, for all $n \in N - \{0\}$

We have **h(1) = 1**, and h(1)= α, so we get $\alpha = 1$

We now use h(n) = **h(n) = 1**, for all $n \in N - \{0\}$ and evaluate

$h(2n) = 3h(n) + \gamma_0 n + \beta_0$	$h(2n+1) = 3h(n) + \gamma_1 n + \beta_1;$
$1 = 3 + \gamma_0 n + \beta_0$	$1 = 3 + \gamma_1 n + \beta_1$
$0 = 2 + \gamma_0 n + \beta_0$	$0 = 2 + \gamma_1 n + \beta_1$
$0 = (2 + \beta_0) + \gamma_0 n$	$0 = (2 + \beta_1) + \gamma_1 n$

We get $\gamma = 0, \quad \beta_0 = \beta_1 = -2$

Solution 1: $\alpha = 1, \quad \gamma = 0, \quad \beta_0 = \beta_1 = -2$

Equation 3

The general form of CF is:

$h(n) = \alpha A(n) + \gamma B(n) + \beta_0 C(n) + \beta_1 D(n)$

We put h(n) = **h(n)** for the **first repertoire function** , i.e. we put h(n) = **h(n) = 1**, for all $n \in N - \{0\}$, i.e.

$\alpha A(n) + \gamma B(n) + \beta_0 C(n) + \beta_1 D(n)$ = **h(n) = 1**, for all $n \in N - \{0\}$, where $\alpha, \gamma,\ \beta_0,\ \beta_1$ already are evaluated in the **Solution 1** as $\alpha = 1, \gamma = 0, \quad \beta_0 = \beta_1 = -2$

We get that CF = RF if and only if the following holds

Fact 3 For all $n \in N - \{0\}$,

$$A(n) - 2C(n) - 2D(n) = 1$$

This is our **Equation 3**

Repertoire Function 2

Consider a **repertoire function 2**: **h(n) = n**, for all $n \in N - \{0\}$

We put h(n) = **h(n) = n**, for all $n \in N - \{0\}$

h(1) = α, **h(1)** = 1 and h(n)=**h(n)**, hence $\alpha = 1$

We now use h(n) = **h(n) = n**, for all $n \in N - \{0\}$ and evaluate

$h(2n) = 3h(n) + \gamma n + \beta_0$	$h(2n+1) = 3h(n) + \gamma n + \beta_1$;
$2n = 3n + \gamma n + \beta_0$	$2n + 1 = 3n + \gamma n + \beta_1$
$0 = (\gamma + 1)n + \beta_0$	$0 = (\gamma + 1)n + (\beta_1 - 1)$

We get $\gamma = -1$, $\beta_0 = 0$, $\beta_1 = 1$ and

Solution 2: $\alpha = 1$, $\gamma = -1$, $\beta_0 = 0$, $\beta_1 = 1$

Equation 4

CF: $h(n) = \alpha A(n) + \gamma B(n) + \beta_0 C(n) + \beta_1 D(n)$

We evaluate CF for h(n) = **h(n) = n**, for all $n \in N - \{0\}$ and for the **Solution 2:** $\alpha = 1, \gamma = -1,\ \beta_0 = 0,\ \beta_1 = 1$ and get

CF = RF if and only if the following holds

Fact 4 For all $n \in N - \{0\}$

$$A(n) - B(n) + D(n) = n$$

This is our **Equation 4**

Repertoire Method: System of Equations

We obtained the following system of **4 equations** on A(n), B(n), C(n), D(n)

1. $\alpha A(n) + \beta_0 C(n) + \beta_1 D(n) = (\alpha, \beta_{b_{m-1}}, ...\beta_{b_0})_3$

2. $A(n) = 3^k$

3. A(n) - 2C(n) - 2D(n) = 1

4. A(n) - B(n) + D(n) = n

We solve it on A(n), B(n), C(n), D (n) and put the solution into

$h(n) = \alpha A(n) + \gamma B(n) + \beta_0 C(n) + \beta_1 D(n)$