cse547, math547
DISCRETE MATHEMATICS

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LECTURE 3

CHAPTER 1
PART FOUR: The Generalized Josephus Problem
Repertoir Method

Josephus Problem Generalization

Our function $J: N - \{0\} \longrightarrow N$ is defined as

J(1) = 1, J(2n) = 2J(n) - 1, J(2n+1) = 2J(n) + 1 for $n > 1$

We generalize it to function $f: N - \{0\} \longrightarrow N$ defined as follows

$$f(1) = \alpha$$

$$f(2n) = 2f(n) + \beta, \quad n \geq 1$$

$$f(2n+1) = 2f(n) + \gamma, \quad n \geq 1$$

Observe that J = f for $\alpha = 1, \beta = -1, \gamma = 1$

NEXT STEP: Find a Closed Formula for f

From RF to CF

Problem: Given RF

$$f(1) = \alpha$$
$$f(2n) = 2f(n) + \beta$$
$$f(2n+1) = 2f(n) + \gamma$$

Find a CF for it

Step 1 Find few initial values for f

Step 2 Find (guess) a CF formula from Step 1

Step 3 Prove correctness of the CF formula, i.e. prove that RF = CF

This step is s usually done by mathematical Induction over the domain of the function f

From RF to CF

Step 1

Evaluate few initial values for

$$f(1) = \alpha$$

$$f(2n) = 2f(n) + \beta$$

$$f(2n+1) = 2f(n) + \gamma$$

Repertoire Method

$n = 2^k + l, \quad 0 \le l < 2^k$

2^0	1	α $\quad$ l = 0	f(1) = α
2^1	2	$2\,\alpha + 1\,\beta + 0\,\gamma \quad 1 = 2^1 - 1 - 0,$ l =0	f(2) = 2f(1) + β $\quad$ l = 0
$2^1 + 1$	3	$2\,\alpha + 0\,\beta + 1\gamma \quad 0 = 2^1 - 1 - 1, \; l = 1$	f(3) = 2f(1) + γ $\quad$ l = 1
2^2	4	$4\,\alpha + 3\beta \quad 3 = 2^2 - 1 - 0$	f(4) = 2f(2) + β $\quad$ l = 0
$2^2 + 1$	5	$4\,\alpha + 2\beta + \gamma \quad 2 = 2^2 - 1 - 1$	f(5) = 2f(2) + γ $\quad$ l = 1
$2^2 + 2$	6	$4\,\alpha + \beta + 2\gamma \quad 2 = l$	f(6) = 2f(3) + β $\quad$ l = 2
$2^2 + 3$	7	$4\,\alpha + 3\gamma \quad 3 = l$	f(7) = 2f(3) + γ $\quad$ l = 3
2^3	8	$8\,\alpha + 7\beta$	F(8) = 2f(4) + β $\quad$ l = 0
$2^3 + 1$	9	$8\,\alpha + 6\beta + 3\gamma$	f(9) = 2f(4) + γ $\quad$ l = 1

Observations

$n = 2^k + l, \quad 0 \le l < 2^k$

α coefficient is 2^k

β coefficient for the groups **decreases** by 1 **down to** 0

β coefficient is $2^k - 1 - l$

γ coefficient **increases** by 1 **up from** 0

γ coefficient is l

General Form of CF

Given a RC function

$$f(1) = \alpha, \quad f(2n) = 2f(n) + \beta, \quad f(2n+1) = 2f(n) + \gamma$$

A general form of CF is

$$f(n) = \alpha A(n) + \beta B(n) + \gamma C(n)$$

for certain $A(n), B(n), C(n)$ to be determined

Our **quess** is:

$$A(n) = 2^k, \quad B(n) = 2^k - 1 - l, \quad C(n) = l$$

for $\qquad n = 2^k + l$

General form of CF

RF: $f(1) = \alpha, \quad f(2n) = 2f(n) + \beta, \quad f(2n+1) = 2f(n) + \gamma$

$CF: \quad f(n) = \alpha A(n) + \beta B(n) + \gamma C(n)$

We **prove** by mathematical Induction over k that when

$n = 2^k + l, \;\; 0 \leq l < 2^k$ our **guess** is true, i.e.

$$A(n) = 2^k, \quad B(n) = 2^k - 1 - l, \quad C(n) = l$$

STEP 1: We consider a case: $\alpha = 1, \beta = \gamma = 0$ and we get

$RF: \quad f(1) = 1, \quad f(2n) = 2f(n), \quad f(2n+1) = 2f(n)$ and

$CF: \quad f(n) = A(n)$

Fact 1

We use $f(n) = A(n)$ and re-write RF in terms of A(n) as follows

$$AR: \quad A(1) = 1, \qquad A(2n) = 2A(n), \quad A(2n+1) = 2A(n)$$

Fact 1 Closed formula CA for AR is:

$$CAR: \quad A(n) = A(2^k + l) = 2^k, \quad 0 \leq l < 2^k$$

Proof by induction on k

Base Case; k=0, i.e n=$2^0 + l$, $0 \leq l < 1$, and we have that n = 1 and evaluate

AR: A(1) = 1, CAR: A(1) = 2^0 = 1, and hence AR = CAR

Fact 1

Inductive Assumption:

$A(2^{k-1}+l) = A(2^{k-1}+l) = 2^{k-1}, \quad 0 \le l < 2^{k-1}$

Inductive Thesis:

$A(2^{k}+l) = A(2^{k}+l) = 2^{k}, \quad 0 \le l < 2^{k}$

Two cases: $n \in even, \quad n \in odd$

C1: $n \in even$

n := 2n, and we have $2^k + l = 2n$ iff $l \in even$

Fact 1

We evaluate n:

$2n = 2^k + l, \quad n = 2^{k-1} + \frac{l}{2}$

We use n in the inductive step

Observe that the **correctness** of using $\frac{l}{2}$ follows from that fact that $l \in even$ so $\frac{l}{2} \in N$ and it can be proved formally like on the previous slides

Proof

$A(2n) =^{reprn} A(2^k + l) =^{evaln} 2A(2^{k-1} + \frac{l}{2}) =^{ind}$
$2 * 2^{k-1} = 2^k$

Fact 1

C2: $n \in odd$

n:= 2n+1, and we have $2^k + l = 2n + 1$ iff $l \in odd$

We evaluate n:

$2n + 1 = 2^k + l, \quad n = 2^{k-1} + \frac{l-1}{2}$

We use n in the inductive step. Observe that the correctness of using $\frac{l-1}{2}$ follows from that fact that $l \in odd$ so $\frac{l-1}{2} \in N$

Proof:

$A(2n+1) =^{reprn} A(2^k + l) =^{evaln} 2A(2^{k-1} + \frac{l-1}{2}) =^{ind}$
$2 * 2^{k-1} = 2^k$

It ends the proof of the **Fact 1**: $A(n) = 2^k$

Repertoire Method

GENERAL PROBLEM

We have a certain recursive formula RF that depends on some parameters, in our case $\alpha, \beta, \gamma,$ i.e.

$RF = RF(n, \alpha, \beta, \gamma)$

We want to find a formula CF of the form

$CF(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$ such that CF = RF

GOAL: find A(n), B(n), C(n) - we have **3 unknowns** so we need **3 equations** to find a solution and then we have to **prove**

$RF(n, \alpha, \beta, \gamma) = A(n)\alpha + B(n)\beta + C(n)\gamma$ for all $n \in N$

In general, when there are k parameters we need to develop and **solve** k equations, and then to **prove**

$RF(n, \alpha_1 ...... \alpha_k) = A_1(n)\alpha_1 + ... + A_k(n)\alpha_k$ for all $n \in N$

Repertoire Method

METHOD: we use a **repertoire** of special functions $\mathbf{R_1} = \mathbf{R_1(n)},\ \mathbf{R_2} = \mathbf{R_2(n)},\ \mathbf{R_3} = \mathbf{R_3(n)}$ and form and solve a system of **6 equations:**

(1) $RF(n, \alpha, \beta, \gamma) = \mathbf{R_i(n)}$, for all $n \in N$, $i = 1, 2, 3$

(2) $CF(n) = A(n)\alpha + B(n)\beta + C(n)\gamma = \mathbf{R_i(n)}$, for all $n \in N$, $i = 1, 2, 3$

For each **repertoire** function $\mathbf{R_i}$ we evaluate corresponding α, β, γ from (1), for $i = 1, 2, 3$

For each **repertoire** function $\mathbf{R_i}$, we put corresponding **solutions** α, β, γ from (1) in (2) to get **3 equations** on A(n), B(n), C(n) and **solve** them on A(n), B(n), C(n)

This also **proves** that $RF(n) = CF(n)$, for all $n \in N$, i.e RF = CF

Repertoire Function R_1

RF: $f(1) = \alpha, \quad f(2n) = 2f(n) + \beta, \quad f(2n+1) = 2f(n) + \gamma$

$CF: \quad f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$

We have already proved in **Step 1** the formula for A(n), so we need only to consider **2 repertoire functions**

Step 2: Consider as the **first repertoire function** $\mathbf{R_1}$ given by a formula

$$\mathbf{R_1(n) = 1} \quad \text{for all} \quad n \in N$$

By (1) $f(n) = \mathbf{R_1(n) = 1}$ for all $n \in N$ i.e. we have the following condition

C1: $f(n) = 1$ for all $n \in N$

By RF we have that $f(1) = \alpha$, and by **C1** : f(1) = 1, and hence $\alpha = 1$

Repertoire Function R_1

RF: $f(1) = \alpha, \quad f(2n) = 2f(n) + \beta, \quad f(2n+1) = 2f(n) + \gamma$

We still consider as the first **repertoire** function given by the formula

$$\mathbf{R_1(n) = 1} \quad \text{for all} \quad n \in N$$

By (1) $f(n) = \mathbf{R_1(n) = 1}$ for all $n \in N$ i.e. we have the following condition

C1: $f(n) = 1$ for all $n \in N$

By RF: $f(2n) = 2f(n) + \beta$ and by **C1** we get equation:

$1 = 2 + \beta$, and hence $\beta = -1$

By RF: $f(2n+1) = 2f(n) + \gamma$ and by **C1** we get equation:

$1 = 2 + \gamma$ and hence $\gamma = -1$

Solution from first **repertoire** function $\mathbf{R_1}$ is

$$\alpha = 1 \quad \beta = -1 \quad \gamma = -1$$

Repertoire Function R_1

Now we use the first **repertoire** function $\mathbf{R_1}$ to the closed formula

$CF: \quad f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$

By (2) we get

$f(n) = \mathbf{R_1} = \mathbf{1}$, for all $n \in N$

We input parameters $\alpha = 1, \ \beta = -1, \ \gamma = -1$ evaluated by RF and $\mathbf{R_1}$ in

(2) $A(n)\alpha + B(n)\beta + C(n)\gamma = \mathbf{R_1(n)} = \mathbf{1}$), for all $n \in N$

and we get the **first equation**

$A(n) - B(n) - C(n) = 1$, for all $n \in N$

By the Repertoire Method we have that CF = RF iff the following holds

FACT 2

A(n) - B(n) - C(n) = 1, for all $n \in N$

Repertoire Function R_2

Step 3:

RF: $f(1) = \alpha$, $f(2n) = 2f(n) + \beta$ $f(2n + 1) = 2f(n) + \gamma$

$CF: \quad f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$

Consider a second **repertoire** function $\mathbf{R_2}$ given by the formula

$$\mathbf{R_2(n) = n} \quad \text{for all} \quad n \in N$$

By (1) $f(n) = \mathbf{R_2(n) = n}$ i.e. we have the following condition

C2: $f(n) = n$, for all $n \in N$

By RF we have that $f(1) = \alpha$, and by **C2** : $f(1) = 1$, and hence $\alpha = 1$

Repertoire Function R_2

RF: f(1) = α,   f(2n) = 2f(n) + β f(2n + 1) = 2f(n) + γ

We still consider as the second **repertoire** function given by the formula

$$\mathbf{R_2(n) = n} \quad \text{for all} \quad n \in N$$

By (1) $f(n) = \mathbf{R_2(n) = n}$ i.e. we have the following condition

C2: $f(n) = n$, for all $n \in N$

By RF: $f(2n) = 2f(n) + \beta$ and by **C2** we get

$2n = 2n + \beta$, and hence $\beta = 0$

By RF: $f(2n + 1) = 2f(n) + \gamma$ and by **C2** we get

$2n + 1 = 2n + \gamma$ and hence $\gamma = 1$

Solution from **second repertoire function** $\mathbf{R_2}$ is

$\alpha = 1$, $\beta = 0$, $\gamma = 1$

Now we use the second **repertoire** function $\mathbf{R_2}$ to the closed formula

$CF: \quad f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$

By (2) we get

$f(n) = \mathbf{R_2} = \mathbf{n}$, for all $n \in N$

We input parameters $\alpha = 1, \ \beta = 0, \ \gamma = 1$ evaluated by RF and $\mathbf{R_2}$ in

(2) $A(n)\alpha + B(n)\beta + C(n)\gamma = \mathbf{R_2(n)} = \mathbf{n}$, for all $n \in N$

and get the **second equation**

A(n) + C(n) = n, for all $n \in N$

By the Repertoire Method we have that CF = RF iff the following holds

FACT 3

A(n) + C(n) = n, for all $n \in N$

Remember: we have proved that A(n) = 2^k, for n = $2^k + l$

so we do not need any more repertoire functions (and equations)

CF for Generalized Josephus

Step 4 A(n), B(n) and C(n) from the following equations

E1 $A(n) = 2^k, \quad n = 2^k + l, \quad 0 \leq l < 2^k$

E2 A(n) - B(n) - C(n) = 1, for all $n \in N$

E3 A(n) + C(n) = n, for all $n \in N$

E3 and **E1** give us that $2^k + C(n) = 2^k + l$, and so

C C(n) = l

From the above and **E2** we get $2^k - l - B(n) = 1$ and so

B B(n) = 2^k - 1 - l

CF for Generalized Josephus

Observe that A, B, C are exact formulas we have guessed and the following holds

Fact 4

$CF: \quad f(n) = 2^k\alpha + (2^k - 1 - l)\beta + l\gamma$ for
$n = 2^k + l, \;\; 0 \leq l < 2^k$

is the closed formula for

RF: $f(1) = \alpha$, $\;f(2n) = 2f(n) + \beta \quad f(2n+1) = 2f(n) + \gamma$

This also ends the proof that Generalized Josephus CF exists and RF = CF

Short CF Solution

Step 2:

RF: $f(1) = \alpha$, $f(2n) = 2f(n) + \beta$ $f(2n+1) = 2f(n) + \gamma$

Here is a **short solution** as presented in our **Book**

You can use it for your problems solutions (also on the tests)- when you really understand what are you doing.

Consider a constant function $f(n) = 1$, for all $n \in N$ (this is our first repertoire function R_1)

We evaluate now α, β, γ for it (if possible)

Solution $1 = 2 + \beta$, $1 = 2 + \gamma$, and so

$$\alpha = 1, \ \beta = -1, \quad \gamma = -1$$

Short CF Solution

$CF: \quad f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$

We evaluate CF for α, β, γ being solutions for RF and f(n) = 1 and get

CF = RF iff the following holds

Fact 2

$$A(n) - B(n) - C(n) = 1 \quad \text{for all} \quad n \in N$$

Short CF Solution

Step 3

RF: $f(1) = \alpha$, $f(2n) = 2f(n) + \beta$ $f(2n+1) = 2f(n) + \gamma$

Consider a constant function $f(n) = n$, for all $n \in N$

We evaluate now α, β, γ for it (if possible)

$2n = 2n + \beta$, $2n + 1 = 2n + \gamma$ and get

Solution: $\alpha = 1, \beta = 0, \gamma = 1$

Short CF Solution

$CF: \quad f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$

Now we evaluate CF for the solutions

$\alpha = 1,\ \beta = 0,\ \gamma = 1$ and f(n) = n

and we get

Fact 3

A(n) + C(n) = n, for all $n \in N$

Final Solution for CF

Step 4

We put together **Facts 1, 2, 3** to evaluate formulas for A(n), B(n), C(n)

Fact 3 and **Fact 1** give that $2^k + C(n) = 2^k + l$, and so

C(n) = l

From the above and **Fact 2** we get $2^k - l - B(n) = 1$ and so

B(n) = 2^k - 1 - l

Final Solution for CF

Given RF, CF defined as follows

RF: $f(1) = \alpha$, $f(2n) = 2f(n) + \beta$ $f(2n + 1) = 2f(n) + \gamma$

$CF: \quad f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$

The final form of CF is as below

Fact 4

$CF: \quad f(n) = 2^k\alpha + (2^k - 1 - l)\beta + l\gamma$, where

$n = 2^k + l, \ \ 0 \leq l < 2^k$

Observe that the **Book does not prove** that CF = RF