# cse547, math547
# DISCRETE MATHEMATICS

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LECTURE 2

## CHAPTER 1
## PART THREE: The Josephus Problem

Flavius Josephus - historian of 1st century

Josephus Story: During Jewish-Roman war he was among 41 Jewish rebels captured by the Romans. They preferred suicide to the capture and decided to form a circle and to kill every third person until no one was left.

Josephus with with one friend wanted none of this suicide nonsense.

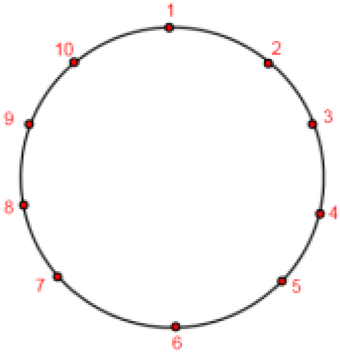
He calculated where he and his friend should stand to avoid being killed and they were saved.

## The Josephus Problem - Our variation

$n$ people around the circle

We **eliminate** each second remaining person until one survives. We denote by $J(n)$ the **position** of a surviver

Example $n = 10$




Elimination order: 2, 4, 6, 8, 10, 3, 7, 1, 9.

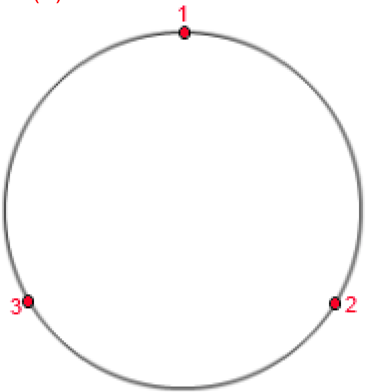
As a result, number 5 survives, i.e. $J(n) = 5$

## Problem: Determine survivor number J(n)

We know that J(n) = 5

We **evaluate** now J(n) for n=1,2,6

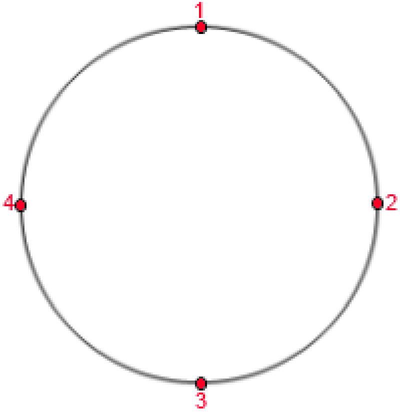
J(1)=1, J(2) = 1, J(3):




We get that J(3)=3

# Determine survivor number J(n)

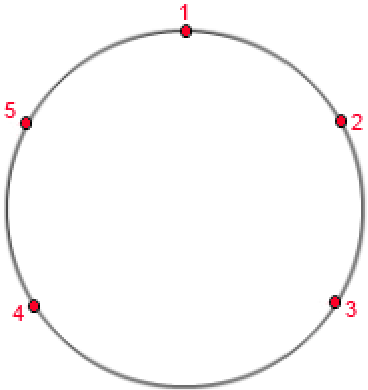
Picture for J(4):




We get J(4)=1

## Determine survivor number J(n)

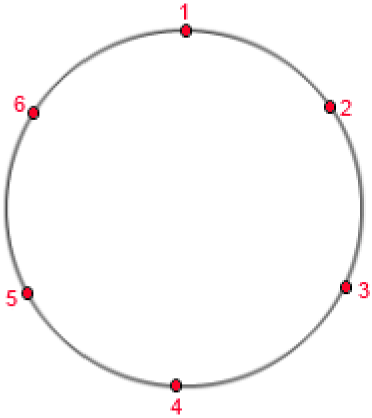
Picture for J(5):




We get J(5)=3

## Problem: Determine survivor number J(n)

Picture for J(6):




We get J(6)=5

## Determine survivor number J(n)

We put our results in a table:

| n | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| J(n) | 1 | 1 | 3 | 1 | 3 | 5 |

**Observation**

All our J(n) after the first run are odd numbers

**Fact**

First trip eliminates all even numbers

Determine survivor number J(n)

**Fact**

First trip eliminates all even numbers

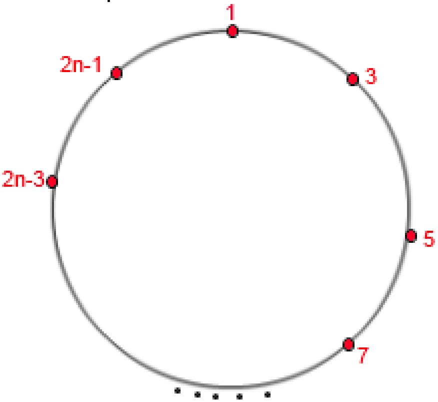
**Observation**

If $n \in$ EVEN we arrive to similar situation we started with half as many people (numbering has changed)

# Determine survivor number J(n)

ASSUME that we START with 2n people

After first trip we have




3 goes out next

This is like starting with n except **each** person has been doubled and decreased by 1

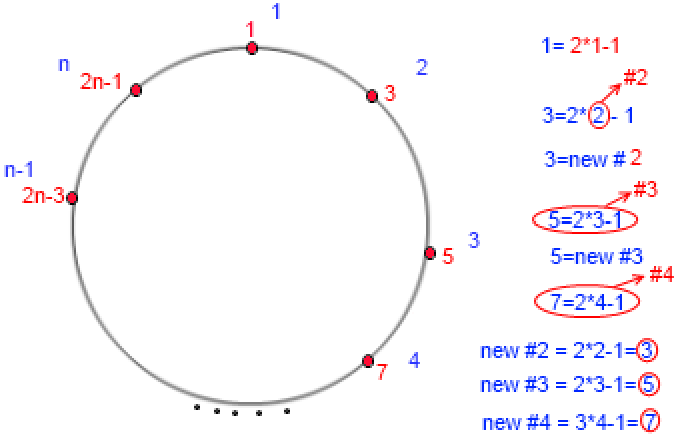
# Determine survivor number J(n)

**Case** n=2n

We get **J(2n)=2J(n) -1** (each person has been doubled and decreased by 1)

We know that J(10)=5, so J(20) = 2J(10)-1 = 2*5-1 = 9

**Re-numbering**

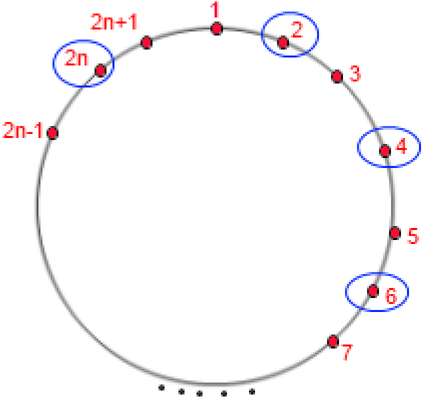


# Determine survivor number J(n)

**Case** n=2n+1

ASSUME that we start with 2n+1 people:

First looks like that

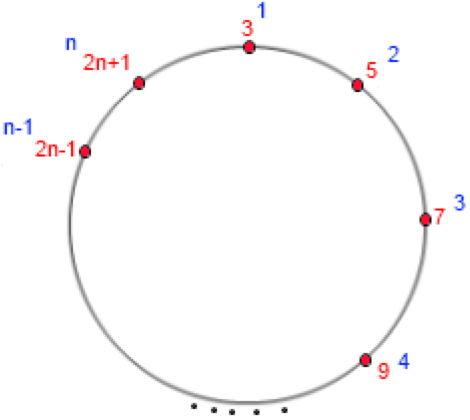



1 is wipped out after 2n

We want to have n-elements after **first** round

## Determine survivor number J(n)

After the first trip we have

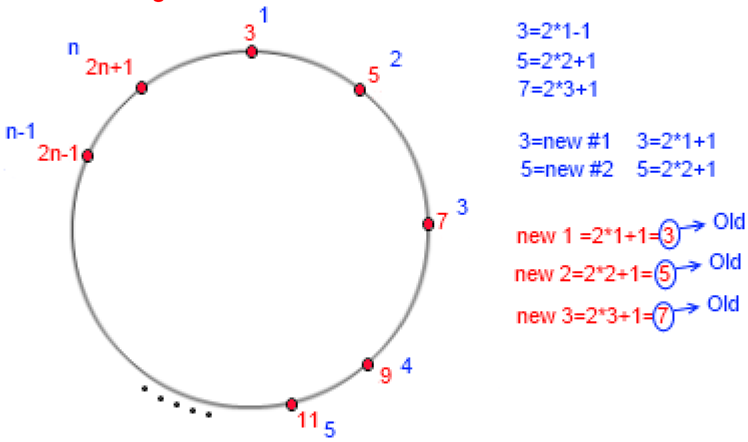



This is like starting with n except that now each person is doubled and increased by 1

# Determine survivor number J(n)

**CASE** n=2n + 1 c.d.

Re-numbering




Formula: new number k = 2k+1

J(2n+1)=new number J(n)

**J(2n+1)=2J(n)+1**

## Recurrence Formula for J(n)

The Recurrence Formula RF for J(n) is:

$J(1) = 1$

$J(2n) = 2J(n) - 1$

$J(2n + 1) = 2J(n) + 1$

Remember that J(k) is a position of the survivor

This formula is more efficient then getting F(n) from F(n-1)

It reduces *n* by factor 2 each time it is applied. We need only 19 application to evaluate $J(10^6)$

## From Recursive Formula to Closed Form Formula

In order to find a **Closed Form Formula** (CF) equivalent to given **Recursive Formula** RF we ALWAYS follow the the Steps 1 - 4 listed below.

Step 1 Compute from recurrence RF a TABLE for some initial values. In our case RF is:
$J(1) = 1,\ \ J(2n) = 2J(n) - 1,\ J(2n+1) = 2J(n) + 1$

Step 2 **Look** for a pattern formed by the values in the TABLE

Step 3 **Find** - guess a closed form formula CF for the pattern

Step 4 **Prove** by Mathematical Induction that RF = CF

# TABLE FOR J(n)

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| J(n) | 1 | 1 | 3 | 1 | 3 | 5 | 7 | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 1 |
| | G1 | G2 | | G3 | | | | G4 | | | | | | | | G5 |

**Observation:** $J(n) = 1$ for $n = 2^k$, $k = 0, 1, ..$

**Next step:** we form groups of J(n) for n consecutive powers of 2 and observe that

| J(n) | G1 | G2 | G3 | G4 | G5 | ... |
|---|---|---|---|---|---|---|
| n | $2^0$ | $2^1 + l$ | $2^2 + l$ | $2^3 + l$ | $2^4 + l$ | ... |

for $0 \leq l < 2^{(k-1)}$ and $k = 1, 2, ...5,$

## Computation of J(n)

**Observe** that for each group $G_k$ the corresponding ns are $n = 2^{k-1} + l$ for all $0 \leq l < 2^{(k-1)}$

and the value of $J(n)$ for $n = 2^k + l$ i.e. $J(n) = J(2^k + l)$ **increases by 2** within the group

Let's now make a TABLE for the group G3

| J(n) | 1 | 3 | 5 | 7 = 2l+1 |
|---|---|---|---|---|
| n | $2^2$ | $2^2 + l$ | $2^2 + 2$ | $2^2 + 3$ |
| | l=0 | l=1 | l=2 | l=3 |

## Guess for CF formula for J(n)

Given $n = 2^{k-1} + l$ we **observed** that $J(n) = 2l + 1$

We **guess** that our CF formula is

$$J(2^k + l) = 2l + 1,$$

for any $k \geq 0, \quad 0 \leq l < 2^k$

## Representation of $n$

$n = 2^k + l$ is called a representation of $n$ when
$l$ is a **remainder** by dividing $n$ by $2^k$ and
$k$ is the largest power of 2 not exceeding $n$

Observe that $2^k \leq n < 2^{k+1}$, $l = n - 2^k$ and so
$0 \leq l < 2^{k+1} - 2^k = 2^m$, i.e.

$$0 \leq l < 2^k$$

## Proof RF = CF

RF: $J(1) = 1, J(2n) = 2J(n) - 1, J(2n+1) = 2J(n) + 1$

CF: $J(2^k + l) = 2l + 1$ , for $n = 2^k + l,\quad k \geq 0, 0 \leq l < 2^k$

**Proof:** by Mathematical Induction on k

**Base Case**: k=0.

Observe that $0 \geq l < 2^0 = 1$, and l = 0 , $n = 2^0 + 0 = 1$, i.e. $n = 1$.

We evaluate $J(1) = 1$, $J(2^0) = 1$ , i.e.

$$RF = CF$$

## Proof RF = CF

**Induction Step** over k has two cases

**c1:** **n** $\in$ **even** and $J(2n) = 2J(n) - 1$

**c2:** **n** $\in$ **odd** and $J(2n+1) = 2J(n) + 1$

**Induction Assumption** for k is

$J(2^{k-1} + l)$ = $2l + 1$, for $0 \leq l < 2^{k-1}$

**case c1:** **n** $\in$ **even**

put n:= 2n, i.e. $2^k + l = 2n,\quad 0 \leq l < 2^k$

Observe that

$2^k + l = 2n$ iff $l \in$ even, i.e. $l = 2m$, and

l/2 = m $\in N$ and $0 \leq \frac{l}{2} < 2^{k-1}$.

## Proof RF = CF

We evaluate $n$ from $2^k + l = 2n$: $n = \frac{2^k + l}{2}$, i.e.

$n = 2^{k-1} + \frac{l}{2}$ , for $0 \leq \frac{l}{2} < 2^{k-1}$, $\frac{l}{2} \in N$

**Proof** in case **c1:** $\mathbf{n} \in$ **even** and $J(2n) = 2J(n) - 1$

$J(2^k + l) =^{reprn} 2J(2^{k-1} + \frac{l}{2}) - 1$

$=^{ind} 2(2\frac{l}{2} + 1) - 1 = 2l + 2 - 1$

$= 2l + 1$.

# Proof RF = CF

**Proof** in case **c2:** **n** $\in$ **odd** and $J(2n+1) = 2J(n)+1$

Inductive Assumption: $J(2^{k-1}+l) = 2l+1$, for $0 \leq l < 2^{k-1}$

Inductive Thesis: $J(2^k+l) = 2l+1$, for $0 \leq l < 2^k$

we put n:=2n+1 and observe that

$2^k + l = 2n+1$ iff l$\in$ odd, i.e.

l= 2m+1, for certain $m \in N$, l-1 =2m, and $\frac{l-1}{2} = m \in N$

# Proof RF = CF

We evaluate, as before $n$ from $2^k + l = 2n + 1$ as follows.

$2^k + (l - 1) = 2n$ and $n = 2^{k-1} + \frac{l-1}{2}$

Proof in case **c2:** **n** $\in$ **odd** and $J(2n+1) = 2J(n) + 1$ is as follows

$J(2^k + l) =^{reprn} 2J(2^{k-1} + \frac{l-1}{2}) + 1$

$=^{ind} 2(2\frac{l-1}{2} + 1) + 1 = 2(l - 1 + 1) + 1$

$= 2l + 1.$

## Some Facts

**Fact 1** $\forall_m J(2^m) = 1$

Proof by induction over m

Observe that $2^m \in Even$, so we use the formula $J(2n) = 2J(n) - 1$, and get

$J(2^m) = J(2 * 2^{m-1} =^{def} 2J(2^{m-1}) - 1 =^{ind} 2 * 1 - 1 = 1$

Hence we also have

**Fact 2** First person will always survive whenever n is a power of 2

# General Case

**Fact 3** Let $n = 2^m + l$ . Observe that the number of people is reduced to power of 2 after there have been $l$ executions.

The first remaining person, the survivor is number $2l + 1$

Our solution

$$J(2^m + l) = 2l + l$$

where $n = 2^m + l$ and $0 \leq l < 2^m$ depends heavily on powers of 2

Let's look now at the binary expansion of n and see how we can **simplify** the computations

## Binary Expansion of n

Definition

$$n = (b_m b_{m-1} ... b_1 b_0)_2$$

stands for

$$n = b_m 2^m + b_{m-1} 2^{m-1} + .. b_1 2 + b_0$$

for

$$b_i \in 0, 1, \quad b_m = 1$$

## Binary Expansion of n

EXAMPLE: n=100

n = $(1\ 1\ 0\ 0\ 1\ 0\ 0)_2$

$2^6 2^5 2^4 2^3 2^2 2^1 2^0$

n = $2^6 + 2^5 + 2^2 = 64 + 32 + 4 + 100$

# Binary Expansion of n

Let now :

n = $2^m + l$, $\quad 0 \leq l < 2^m$

we have the following binary expansions:

1) $l = (0, b_{m-1}, .., b_1, b_0)_2$ as $l < 2^m$

2) ]$2l = (b_{m-1}, .., b_1, b_0, 0)_2$ as

$$l = b_{m-1}2^{m-1} + .. + b_1 2 + b_0$$

$$2l = b_{m-1}2^m + .. + b_1 2^2 + b_0 2 + 0$$

3) $2^m = (1, 0, ..., 0)_2, 1 = (0...1)_2$

4) $n = 2^m + l$

$$n = (1, b_{m-1}, .., b_1, b_0)_2 \;\; \textit{from} \;\; 1 + 3$$

5) $2l + 1 = (b_{m-1}, b_{m-2}, .., b_0, 1)_2 \quad$ from $2 + 3$

# Binary Expansion Josephus

CF : $J(n) = 2l + 1,$
for $n = 2^m + l$

From 5 on last slide we can re-write CF in binary expansion as follows

$$BF: \quad J((b_m, b_{m-1}, .., b_1, b_0)_2) = (b_{m-1}, .., b_1, b_0, b_m)_2$$

because $b_m = 1$ in binary expansion of n, we get

$$BF: \quad J((1, b_{m-1}, .., b_1, b_0)_2) = (b_{m-1}, .., b_1, b_0, 1)_2$$

## Binary Expansion Josephus

Example: Find J(100)

$n = 100 = (1100100)_2$

$$J(100) = J((1100100)_2) =^{BF} (1001001)_2$$

J(100) = 64 + 8 + 1 = 73

$$BF: \quad J((1, b_{m-1}, .., b_1, b_0)_2) = ((b_{m-1}, .., b_1, b_0, 1)_2$$

## Josephus Generalization

Our function $J: \; N - \{0\} \longrightarrow N$ is defined as

J(1) = 1, J(2n) = 2J(n) - 1, J(2n+1) = 2J(n) + 1 for $n > 1$

We generalize it to function $f: \; N - \{0\} \longrightarrow N$ defined as follows

$$f(1) = \alpha$$

$$f(2n) = 2f(n) + \beta, \qquad n \geq 1$$

$$f(2n+1) = 2f(n) + \gamma, \qquad n \geq 1$$

Observe that J = f for $\alpha = 1, \beta = -1, \gamma = 1$

NEXT STEP: Find a Closed Formula for f