cse547, math547
DISCRETE MATHEMATICS

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LECTURE 10

CHAPTER 2
SUMS

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Part 5: Infinite Sums- Infinite Series - Lecture 10

Infinite Sums (Series)

We **extend** now the notion of a **finite sum** $\Sigma_{k=1}^{n} a_k$ to an **infinite sum**

$$\Sigma_{n=1}^{\infty} a_n$$

For a given a sequence $\{a_n\}_{n \in N-\{0\}}$, i.e the sequence

$$a_1, a_2, a_3, ...a_n, ..$$

we consider a following (infinite) **sequence**

$S_1 = a_1, \quad ...., \quad S_n = \Sigma_{k=1}^{n} a_k, \quad S_{n+1} = \Sigma_{k=1}^{n+1} a_k, \quad ....$

and define the **infinite sum** as follows

Infinite Sum Definition

Definition 1

If the limit of the sequence $\{S_n = \Sigma_{k=1}^{n} a_k\}_{n \in N-\{0\}}$ **exists** we call it an **infinite sum** of the sequence $\{a_n\}_{n \in N-\{0\}}$

We write it as

$$\Sigma_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \Sigma_{k=1}^{n} a_k$$

The sequence $\{S_n\}_{n \in N-\{0\}}$ is called its **sequence of partial sums**

Infinite Sum Definition

Definition 2

If the limit $\lim_{n\to\infty} S_n$ **exists** and **is finite**, i.e.

$$\lim_{n\to\infty} S_n = S$$

then we say that the **infinite sum** $\Sigma_{n=1}^{\infty} a_n$ **converges** to S and we write

$$\Sigma_{n=1}^{\infty} a_n = \lim_{n\to\infty} \Sigma_{k=1}^{n} a_k = S$$

otherwise the **infinite sum** **diverges**

Observation

Observation 1

In a case when all elements of the sequence

$$\{a_n\}_{n \in N - \{0\}}$$

are equal 0 starting from a certain $k \geq 1$

the **infinite sum** $\Sigma_{n=1}^{\infty} a_n$ becomes a **finite sum**

The **infinite sum** is a generalization of the **finite one**, and this is why we keep the similar notation

Example 1

The infinite sum of a geometric sequence $a_n = x^k$ for $x \geq 0$, i.e. the sum

$\Sigma_{n=1}^{\infty} x^n$ **converges** if and only if $|x| < 1$

It is true because

$\Sigma_{k=1}^{n} x^k = S_n = \frac{x - x^{n+1}}{1-x} = \frac{x(1-x^n)}{1-x}$ and

$$\lim_{n\to\infty} \frac{x(1-x^n)}{1-x} = \lim_{n\to\infty} \frac{x}{1-x}(1-x^n) = \frac{x}{1-x} \quad \text{iff} \quad |x| < 1$$

Moreover

$$\Sigma_{n=1}^{\infty} x^k = \frac{x}{1-x}$$

More Examples

Example 2

The series $\Sigma_{n=1}^{\infty} 1$ **diverges** to ∞ as

$$S_n = \Sigma_{k=1}^{n} 1 = n$$

and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty$$

More Examples

Example 3

The infinite sum $\Sigma_{n=0}^{\infty} (-1)^n$ **diverges**

Proof

We use the Perturbation Method

$$S_n + a_{n+1} = a_0 + \sum_{k=0}^{n} a_{k+1}$$

to eveluate

$$S_n = \Sigma_{k=0}^{n} (-1)^k = \frac{1+(-1)^n}{2} = \frac{1}{2} + \frac{(-1)^n}{2}$$

and we prove that

$$\lim_{n\to\infty} \left(\frac{1}{2} + \frac{(-1)^n}{2}\right)$$ **does not exist**

More Examples

Example 4

The **infinite sum** $\Sigma_{n=0}^{\infty}\frac{1}{(k+1)(k+2)}$ **converges** to 1; i.e.

$$\Sigma_{n=0}^{\infty}\frac{1}{(k+1)(k+2)} = 1$$

Proof: first we evaluate $S_n = \Sigma_{k=0}^{n}\frac{1}{(k+1)(k+2)}$ as follows

$$S_n = \Sigma_{k=0}^{n}\frac{1}{(k+1)(k+2)} = \Sigma_{k=0}^{n}k^{\underline{-2}} = \Sigma_{k=0}^{n+1}k^{\underline{-2}}\ \delta k$$

$$= -\frac{1}{k+1}\Big|_0^{n+1} = -\frac{1}{n+2} + 1$$

and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} -\frac{1}{n+2} + 1 = 1$$

Definition

Definition 3

For any **infinite sum** (series)

$$\Sigma_{n=1}^{\infty} a_n$$

a sum (series)

$$r_n = \Sigma_{m=n+1}^{\infty} a_m$$

is called its n-th **remainder**

Fact 1

If the infinite sum $\Sigma_{n=1}^{\infty} a_n$ **converges**,
then **so does** its n-th **remainder** $r_n = \Sigma_{m=n+1}^{\infty} a_m$

Proof:

Assume that $\Sigma_{n=1}^{\infty} a_n$ **converges**

Let's denote $S_n = \Sigma_{m=1}^{n} a_m$ and we have that

$S = \lim_{n\to\infty} S_n = \Sigma_{m=1}^{\infty} a_m$

Observe that $r_n = S - \Sigma_{m=1}^{n} a_m = S - S_n$

By definition, r_n **converges** iff $\lim_{n\to\infty} r_n$ **exists** and is **finite.**

We evaluate

$\lim_{n\to\infty} r_n = S - \lim_{n\to\infty} S_n = S - S = 0$

what **ends** the proof

General Properties of Infinite Sums

Theorem 1

Theorem 1

If the infinite sum

$$\Sigma_{n=1}^{\infty} a_n \quad \text{converges,} \quad \text{then} \lim_{n\to\infty} a_n = 0$$

Proof: observe that $a_n = S_n - S_{n-1}$ and hence

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = 0$$

as $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1}$

Theorem 1

Remark 1

The **reverse** statement to the **Theorem 1**, namely a statement

$$\text{If} \quad \lim_{n\to\infty} a_n = 0 \quad \text{then} \quad \Sigma_{n=1}^{\infty} a_n \quad \text{converges}$$

is **not always true** as there are infinite sums with the term converging to zero that are not convergent

Observe that **Theorem 1** can be **re-written** as follows

Theorem 1

$$\text{If} \quad \lim_{n\to\infty} a_n \neq 0 \quad \text{then} \quad \Sigma_{n=1}^{\infty} a_n \quad \text{diverges}$$

Example 5

The **infinite harmonic sum** $H = \Sigma_{n=1}^{\infty} \frac{1}{n}$

DIVERGES to ∞, even if its -th term **converges to 0**, i.e.

$\Sigma_{n=1}^{\infty} \frac{1}{n} = \infty$ and $\lim_{n \to \infty} \frac{1}{n} = 0$

The infinite harmonic sum provides an **example** of an infinite diverging sum $\Sigma_{n=1}^{\infty} a_n$, such that $\lim_{n \to \infty} a_n = 0$

Properties

Definition 4

Infinite sum

$$\Sigma_{n=1}^{\infty} a_n$$

is **bounded** if its sequence of partial sums

$$S_n = \Sigma_{k=1}^{n} \; a_k$$

is bounded; i.e.

there is a number $M \in R$ such that $S_n < M$, for all $n \in N$

Fact 2

Every convergent infinite sum is bounded

Properties

Theorem 2

If the infinite sums

$$\Sigma_{n=1}^{\infty} a_n, \qquad \Sigma_{n=1}^{\infty} b_n \qquad \textbf{converge}$$

then the following properties hold.

$$\Sigma_{n=1}^{\infty}(a_n + b_n) = \Sigma_{n=1}^{\infty} a_n \ + \ \Sigma_{n=1}^{\infty} b_n,$$

and

$$\Sigma_{n=1}^{\infty} c a_n = c \Sigma_{n=1}^{\infty} a_n, \quad c \in R$$

Alternating Infinite Sums

Definition 5
An infinite sum

$$\Sigma_{n=1}^{\infty}(-1)^{n+1}a_n, \quad \text{for} \quad a_n \geq 0$$

is called **alternating infinite sum** (alternating series)

Example 6
Consider

$$\Sigma_{n=1}^{\infty}(-1)^{n+1} = 1 - 1 + 1 - 1 + .$$

If we group the terms in pairs, we get

$$(1-1)+(1-1)+.... = 0$$

but if we start the pairing one step later, we get

$$1-(1-1)-(1-1)-..... = 1-0-0-0-... = 1$$

Alternating Series

The **Example 6** shows that grouping terms in a case of infinite sum can lead to inconsistencies (contrary to the finite case)

Look also example on page 59 of our BOOK

We need to develop some strict criteria for manipulations and convergence/divergence of **alternating series**

Alternating Series Theorem

Theorem 3
Given an alternating infinite sum

$$\Sigma_{n=1}^{\infty}(-1)^{n+1}a_n$$

such that

1. $a_n \geq 0$, for all n

2. sequence $\{a_n\}$ is **decreasing**. i.e.

$a_1 \geq a_2 \geq a_3 \geq ....$

3. $\lim_{n\to\infty} a_n = 0$

Then the sum $\Sigma_{n=1}^{\infty}(-1)^{n+1}a_n$ **converges**, i.e.

$$\Sigma_{n=1}^{\infty}(-1)^{n+1}a_n = S$$

Moreover the partial sums $S_n = \Sigma_{k=1}^{n}(-1)^{k+1}a_k$ fulfill the condition

$$S_{2n} \leq \Sigma_{n=1}^{\infty}(-1)^{n+1}a_n \leq S_{2n+1}$$

for all $n \in N^+$

Alternating Series Theorem Proof

Proof

Evaluate

$$S_{2(n+1)} = S_{2n+2} = \Sigma_{k=1}^{2n+2}(-1)^{k+1}a_k$$

$$= \Sigma_{k=1}^{2n}(-1)^{k+1}a_k + (-1)^{2n+2}a_{2n+1} + (-1)^{2n+3}a_{2n+2}$$

$$= S_{2n} + (a_{2n+1} - a_{2n+2})$$

By **2.** we know that sequence $\{a_n\}$ is decreasing hence $a_{2n+1} - a_{2n+2} \geq 0$ and so

$$S_{2n+2} \geq S_{2n}$$

i.e we **proved** that the sequence of S_{2n} is **increasing**

Alternating Series Theorem Proof

We are going to prove now that the sequence of S_{2n} is also **bounded**

Observe that

$$S_{2n} = a_1 - a_2 + a_3 - a_4 + + (-1)^{2n+1} a_{2n}$$

$$= a_1 - (a_2 - a_3) - (a_4 - a_5) + \dots - a_{2n}$$

By **2.** $a_k - a_{k+1} \geq 0$ for $k = 2, 3, \dots, 2(n-1)$ and by **1.** $a_{2n} \geq 0$, so $-a_{2n} \leq 0$ and we get that

$$S_{2n} \leq a_1$$

what proves that S_{2n} is **bounded**

Alternating Series Theorem Proof

We know that any bounded and increasing sequence is is convergent, so we **proved** that S_{2n} **converges**

Let denote $\lim_{n\to\infty} S_{2n} = g$

To prove that

$$\Sigma_{n=1}^{\infty}(-1)^{n+1}a_n = \lim_{n\to\infty} S_n$$

converges we have to show now that also

$$\lim_{n\to\infty} S_{2n+1} = g$$

Observe that $S_{2n+1} = S_{2n} + a_{2n+1}$ and we get

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1} = g$$

as we assumed in **3.** that $\lim_{n\to\infty} a_n = 0$

Alternating Series Theorem Proof

We proved that the sequence S_{2n} is creasing

We prove, in a similar way (exercise!) that the sequence $\{S_{2n+1}\}$ is decreasing

Hence

$$S_{2n} \leq \lim_{n\to\infty} S_{2n} = g = \Sigma_{n=1}^{\infty}(-1)^{n+1}a_n$$

and

$$S_{2n+1} \geq \lim_{n\to\infty} S_{2n+1} = g = \Sigma_{n=1}^{\infty}(-1)^{n+1}a_n$$

what means that

$$S_{2n} \leq \Sigma_{n=1}^{\infty}(-1)^{n+1}a_n \leq S_{2n+1}$$

It **ends the proof** of the **Theorem 3**

Example 7

Consider the ANHARMONIC series (infinite sum)

$$AH = \Sigma_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}..$$

Observe that $a_n = \frac{1}{n} \geq 0$, $\frac{1}{n} \geq \frac{1}{n+1}$ i.e. $a_n \geq a_{n+1}$, for all n, and $\lim_{n\to\infty} a_n = 0$

So the assumptions of the Theorem 3 are fulfilled for AH and hence AH **converges**

In fact, it is proved (by analytical methods, not ours) that

$$AH = \Sigma_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \ln 2$$

Example

A series (infinite sum)

$$\Sigma_{n=0}^{\infty}(-1)^n \frac{1}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}$$

converges by **Theorem 3**

Proof is similar to the one in the **Example 7**

It also is proved (by analytical methods, not ours) that

$$\Sigma_{n=0}^{\infty}(-1)^n \frac{1}{2n+1} = \frac{\pi}{4}$$

and hence we have that

$$\pi = \Sigma_{n=0}^{\infty}(-1)^n \frac{4}{2n+1}$$

Generalization of Theorem 3

Theorem 4 ABEL Theorem

IF a sequence $\{a_n\}$ fulfils the assumptions of the **Theorem 3** i.e.

1. $a_n \geq 0$, for all n

2. sequence $\{a_n\}$ is **decreasing**, i.e.

$a_1 \geq a_2 \geq a_3 \geq ....$

3. $\lim_{n\to\infty} a_n = 0$

and an infinite sum (converging or diverging)

4. $\Sigma_{n=1}^{\infty} b_n$ **is bounded**,

THEN the infinite sum

$$\Sigma_{n=1}^{\infty} a_n b_n$$

always **converges.**

Observe that **Theorem 3** is a special case of **Theorem 4** when $b_n = (-1)^{n+1}$

Convergence of Infinite Sums
with Positive Terms

Infinite Sums with Positive Terms

We consider now infinite sums with all its terms being **positive** real numbers, i.e.

$$S = \sum_{n=1}^{\infty} a_n$$

for

$$a_n \geq 0, \quad a_n \in R$$

Observe that if all $a_n \geq 0$, then the sequence $\{S_n\}$ of **partial sums** $S_n = \Sigma_{k=1}^{n} a_k$ is **increasing**, i.e.

$$S_1 \leq S_2 \leq .... \leq S_n$$

and hence the $\lim_{n \to \infty} S_n$ **exists** and is finite or is ∞

Infinite Sums with Positive Terms

We have just **proved** the following theorem

Theorem 5

The infinite sum

$$S = \Sigma_{n=1}^{\infty} a_n, \quad \text{for} \quad a_n \geq 0, \quad a_n \in R$$

always **converges**, or **diverges** to ∞

Comparing the Series with Positive Terms

Theorem 6 Comparing the series

Let $\Sigma_{n=1}^{\infty} a_n$ be an infinite sum and $\{b_n\}$ be a sequence such that

$$0 \leq b_n \leq a_n \quad \text{for all n}$$

If the infinite sum $\Sigma_{n=1}^{\infty} a_n$ **converges**

then $\Sigma_{n=1}^{\infty} b_n$ also **converges** and

$$\Sigma_{n=1}^{\infty} b_n \leq \Sigma_{n=1}^{\infty} a_n$$

Application of the **Theorem 6**: we can prove the **convergence** of a series $\Sigma_{n=1}^{\infty} b_n$ by bounding the sequence b_n by a certain sequence a_n such that $0 \leq b_n \leq a_n$ and we know that $\Sigma_{n=1}^{\infty} a_n$ **converges**

Proof of Theorem 6

Proof

Let us denote

$$S_n = \Sigma_{k=1}^{n} a_k, \qquad T_n = \Sigma_{k=1}^{n} b_k$$

As $0 \leq b_n \leq a_n$ we get that $T_n \leq S_n$

But we know that the series S_n **converges**, hence

$$S_n \leq \lim_{n\to\infty} S_n = \Sigma_{n=1}^{\infty} a_n = S$$

So we get that

$$T_n \leq S_n \leq \leq \lim_{n\to\infty} S_n = \Sigma_{n=1}^{\infty} a_n = S$$

Proof of Theorem 6

The inequality

$$T_n \leq S$$

means that the sequence $\{T_n\}$ is a bounded sequence (by S) with positive terms, hence the sequence $T_n = \Sigma_{k=1}^{n} b_k$ **converges**, i.e.

$$\lim_{n\to\infty} T_n = T = \Sigma_{n=1}^{\infty} b_n$$

We hence **proved** that the series $\Sigma_{n=1}^{\infty} b_n$ **converges**
But we have also proved that $T_n \leq S_n$, hence

$$\lim_{n\to\infty} T_n \leq \lim_{n\to\infty} S_n$$

which means that

$$\Sigma_{n=1}^{\infty} b_n \ \leq \ \Sigma_{n=1}^{\infty} a_n$$

what **ends the proof**

Example

Example 9

Use **Theorem 6** to prove that the series,

$$\Sigma_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

converges

We prove by analytical methods that it converges to $\frac{\pi^2}{6} - 1$, i.e.

$$\Sigma_{n=1}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6} - 1$$

Here we prove only that it **does converge**

Example 9 Solution

First observe that the series below **converges to 1**, i.e.

$$\Sigma_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Consider

$$\begin{aligned} S_n &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} \cdots + \frac{1}{n(n+1)} \\ &= (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \ldots (\frac{1}{n} - \frac{1}{n+1}) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

so we get

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n \to \infty} S_n = \lim_{n \to \infty} (1 - \frac{1}{n+1}) = 1$$

Example 9 Solution

Now we observe (easy to prove) that

$$\frac{1}{2^2} \leq \frac{1}{1\cdot 2}, \quad \frac{1}{3^2} \leq \frac{1}{1\cdot 3}, \ \ldots\ldots \ \frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)},$$

i.e. we proved that all assumptions of **Theorem 6** hold, hence $\Sigma_{n=1}^{\infty} \frac{1}{(n+1)^2}$ **converges** and moreover

$$\Sigma_{n=1}^{\infty} \frac{1}{(n+1)^2} \leq \Sigma_{n=1}^{\infty} \frac{1}{n(n+1)}$$

and

$$\Sigma_{n=1}^{\infty} \frac{1}{(n+1)^2} \leq 1$$

Theorem 7 **D'Alambert's Criterium**

If $a_n \geq 0$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$

then the series $\sum_{n=1}^{\infty} a_n$ **converges**

Proof of D'Alambert's Criterium

Proof

Let h be any number such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < h < 1$$

It means that there is k such that for any $n \geq k$ we have,

$$\frac{a_{n+1}}{a_n} < h, \qquad \text{i.e.} \qquad a_{n+1} < a_n h$$

Hence,

$a_{k+1} < a_k h, \quad a_{k+2} = a_{k+1} h < a_k h^2,$
$a_{k+3} < a_k h^3, \quad a_{k+4} < a_k h^4, \quad a_{k+5} < a_k h^5, , \ldots$

Proof of D'Alambert's Criterium

We have that all terms a_n of $\Sigma_{n=k}^{\infty} a_n$ are smaller than the terms of a **converging** (as $0 < h < 1$) geometric series

$$\Sigma_{n=0}^{\infty} a_k h^n = a_k + a_k h + a_k h^2 + \ldots$$

By **Theorem 6**, the series

$$\Sigma_{n=1}^{\infty} a_n$$

also **converges**

Cauchy's Criterium

Theorem 8 **Cauchy's Criterium**

If $a_n \geq 0$ and $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$

then the series $\sum_{n=1}^{\infty} a_n$ **converges**

Proof: We carry the proof in a similar way as the proof of D'Alambert Criterium

Proof of Cauchy's Criterium

Let h be any number such that

$$\lim_{n\to\infty} \sqrt[n]{a_n} < h < 1$$

It means that there is k, such that for any $n \geq k$ we have $\sqrt[n]{a_n} < h$ i.e. $a_n < h^n$

This indicates that all terms a_n of $\sum_{n=k}^{\infty} a_n$ are smaller

then the terms of a **converging** (as $0 < h < 1$) geometric series

$$\Sigma_{n=k}^{\infty} h^n = h^k + h^{k+1} + h^{k+2} + \dots$$

By **Theorem 6** the series

$$\Sigma_{n=1}^{\infty} a_n$$

must **converge**

Divergence Criteria

Theorem 9 **Divergence Criteria**

If $a_n \geq 0$ and $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} > 1$ or $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$

then the series $\sum\limits_{n=1}^{\infty} a_n$ **diverges**

Proof of Divergence Criteria

Proof:

Assume that, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$

Then for sufficiently large n we have that

$$\frac{a_{n+1}}{a_n} > 1 \text{ and hence } a_{n+1} > a_n$$

This means that a_n is **strictly increasing** sequence of positive numbers, so $\lim_{n\to\infty} a_n \neq 0$

By **Theorem 1** the series $\sum_{n=1}^{\infty} a_n$ **diverges**

Theorem 1 says: if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

Proof of Divergence Criteria

Similarly, if $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$

then for sufficiently large n, we have that

$$\sqrt[n]{a_n} > 1 \text{ and hence } a_n > 1$$

So it must be that $\lim_{n\to\infty} a_n \neq 0$

By **Theorem 1** the series $\sum_{n=1}^{\infty} a_n$ **diverges**

Theorem 1 says: if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

Convergence/Divergence

Table: Convergence/Divergence for $\sum_{n=1}^{\infty} a_n$

Cauchy Criterium	D'Alembert's Criterium	Convergence/Divergen
$\lim_{n\to\infty} \sqrt[n]{a_n} < 1$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$	Converges
$\lim_{n\to\infty} \sqrt[n]{a_n} > 1$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$	Diverges

Convergence/Divergence

Remark

It can happen that for a certain infinite sum $\sum_{n=1}^{\infty} a_n$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 = \lim_{n\to\infty} \sqrt[n]{a_n}$$

In this case our Divergence Criteria **do not decide** whether the infinite sum **converges** or **diverges**

We say in this case that that the infinite sum does not react on the criteria

There are other, stronger criteria for **convergence** and **divergence**

Example 10

The Harmonic series $H = \sum_{n=1}^{\infty} \frac{1}{n}$ **does not react** on D'Alambert's Criterium (Theorem 7)

Proof: Consider

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})} = 1$$

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ we say , that the **Harmonic series**

$$H = \sum_{n=1}^{\infty} \frac{1}{n}$$

does not react on D'Alambert's criterium

Examples

Example 11

The series $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ **does not react** on D'Alambert's Criterium (Theorem 7)

Proof:

Consider, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{(n+2)^2}$$

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{n^2+4n+1} = \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{1+\frac{4}{n}+\frac{4}{n^2}} = 1$$

Since, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ we say , that the series

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

does not react on D'Alambert's criterium

Other Criteria

Remark

Both series

$$\Sigma_{n=1}^{\infty}\frac{1}{n} \qquad \text{and} \qquad \Sigma_{n=1}^{\infty}\frac{1}{(n+1)^2}$$

do not react on D'Alambert's Criterium

but first series is **divergent** and the second is **convergent**

There are more criteria for convergence

Most known are Kumer's criterium and Raabe criterium

Infinite Sums (Series)
EXAMPLES

Example 1

$$\sum_{n=1}^{\infty} \frac{c^n}{n!} \quad converges \quad for \quad c > 0$$

$HINT: \; Use \; D'Alembert$

Proof:

$$\frac{a_{n+1}}{a_n} = \frac{c^{n+1}}{c^n} \frac{n!}{(n+1)!}$$

$$= \frac{c}{n+1}$$

Example 1

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{c}{n+1}$$

$$= 0 < 1$$

By D'Alembert's Criterium

$$\sum_{n=1}^{\infty} \frac{c^n}{n!}$$ **converges**

Example 2

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} \qquad \textit{converges}$$

Proof:

$$a_n = \frac{n!}{n^n}$$

$$a_{n+1} = \frac{n!(n+1)}{(n+1)^{n+1}}$$

$$\frac{a_n+1}{a_n} = \frac{n! \;\; n^{(n+1)}}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= (n+1) \cdot \frac{n^n}{(n+1)^{n+1}}$$

Example 2

$$
\begin{aligned}
(n+1)^{n+1} &= (n+1)^n \ (n+1) \\
\frac{a_n+1}{a_n} &= \frac{(n+1)\ n^n}{(n+1)^n \ (n+1)} \\
&= (\frac{n}{n+1})^n \\
&= \frac{1}{(1+\frac{1}{n})^n}
\end{aligned}
$$

Example 2

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n}$$

$$= \frac{1}{e} < 1$$

By D'Alembert's Criterium the series,

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$ **converges**

Exercise 1

Prove that

$$\lim_{n\to\infty} \frac{c^n}{n!} = 0 \qquad \textit{for} \quad c > 0$$

Solution:

We have proved in **Example 1**

$$\sum_{n=1}^{\infty} \frac{c^n}{n!} \quad \text{converges} \quad \text{for} \quad c > 0$$

Exercise 1

Theorem 1 says:

$$\text{IF } \sum_{n=1}^{\infty} a_n \quad \text{converges} \quad \text{THEN} \quad \lim_{n\to\infty} a_n = 0$$

Hence by **Example 1** and **Theorem 1** we have proved that

$$\lim_{n\to\infty} \frac{c^n}{n!} = 0 \quad \textit{for} \quad c > 0$$

Observe that we have also proved that n! grows faster than c^n

Exercise 2

Prove that

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

Hint : COMPLICATE IT!

Proof

By **Example 2** we know that

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} \quad \textit{converges}$$

Exercise 2

Theorem 1 says:

$$\text{IF} \quad \sum_{n=1}^{\infty} a_n \quad \text{converges} \quad \text{THEN} \quad \lim_{n\to\infty} a_n = 0$$

Hence,

$$\lim_{n\to\infty} \frac{n!}{n^n} \quad = \quad 0$$

Example 3 Harmonic Series

$$H = \sum_{n=1}^{\infty} \frac{1}{n}$$

does not react on D'Alembert Criterium

Proof

$$\frac{a_{n+1}}{a_n} = \frac{1}{n+1} \frac{n}{1} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$$

Example 4

$$\lim_{n\to\infty} \frac{c^n}{n!} = 0, \qquad \lim_{n\to\infty} \frac{n!}{n^n} = 0$$

Proof: From **Example 1** and D'Alembert's Criteriumwe know that

$$\sum_{n=1}^{\infty} \frac{c^n}{n!} \quad \textit{converges}$$

Example 4

By **Example 2** and D'Alembert's Criterium we have that

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} \quad \textit{converges}$$

By **Theorem 1**

$$\lim_{n\to\infty} \frac{c^n}{n!} = 0, \qquad \lim_{n\to\infty} \frac{n!}{n^n} = 0$$

Example 5

We know that the **Harmonic Series**

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \textbf{diverges}$$

Use this information and Cauchy Criterium **to prove** that,

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

Example 5

Proof Sequence

$a_n = \sqrt[n]{n}$ is for large n decreasing and

$a_n > 1$

Hence

$\lim_{n\to\infty} a_n$ exists and

$$\lim_{n\to\infty} \sqrt[n]{n} \geq 1$$

Example 5

Assume

$$\lim_{n\to\infty} \sqrt[n]{n} > 1 \qquad \text{we get}$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} < 1$$

Cauchy Criterium says:

$$\text{IF} \quad \lim_{n\to\infty} \sqrt[n]{a_n} < 1 \quad \text{THEN}$$

$$\sum_{n=1}^{\infty} a_n \quad \text{converges for} \quad a_n \geq 0, \; a_n \in R$$

Example 5

Hence by Cauchy Criterium

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \textit{converges}$$

This is a **contradiction**, as we know that the Harmonic Series **diverges**

Hence

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

Example 6

We are going to show that the series

$$\sum_{n=1}^{\infty} \frac{|x(x-1)....(x-n+1)|}{n!} \; c^n$$

converges for $0 < c < 1$ and $x \in R$

Example 6

Proof we evaluate

$$\frac{a_{n+1}}{a_n} = \frac{|x(x-1)....(x-n)| \; \cancel{c}^n c}{\cancel{n}!(n+1)} \quad \frac{\cancel{n}!}{|x(x-1)....(x-n+1)| \; \cancel{c}^n}$$

$$= \frac{|x-n|}{n+1} \; c = \frac{|\frac{x}{n} - 1|}{1 + \frac{1}{n}} \; c$$

and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = c$$

Example 6

Hence, by D'Alambert Criterium the series

$$\sum_{n=1}^{\infty} \frac{|x(x-1)....(x-n+1)|}{n!} c^n$$

converges for $0 < c < 1$ and $x \in R$

Example 7

Prove that

$$\lim_{n\to\infty} \frac{|x(x-1)...(x-n+1)|}{n!} c^n \quad = \quad 0 \qquad\qquad 0 < |c| < 1$$

Solution By **Example 6**, the series

$$\sum_{n=1}^{\infty} \frac{|x(x-1)....(x-n+1)|}{n!} \; c^n$$

converges for $0 < c < 1$ and $x \in R$

Theorem 1 says:

IF $\sum_{n=1}^{\infty} a_n$ converges THEN $\lim_{n\to\infty} a_n = 0$

Hence proved

Absolute and Conditional Convergence

Absolute Convergence

Definition

$\sum_{n=1}^{\infty} a_n$ **converge absolutely** iff $\sum_{n=1}^{\infty} |a_n|$ converges

Conditional Convergence

Definition

$$\sum_{n=1}^{\infty} a_n \quad \textbf{converges} \quad \textbf{conditionally}$$

if and only if

$$\sum_{n=1}^{\infty} a_n \quad \text{converges, but } \textbf{not} \quad \textbf{absolutely}$$

i.e. when

$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \quad \text{and} \qquad \sum_{n=1}^{\infty} |a_n| \quad \text{does not converge}$$

Theorem

Theorem 10

IF $\sum_{n=1}^{\infty} a_n$ converges absolutely , THEN it converges

Moreover

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|$$

Examples

Example 8

Geometric series

$$\sum_{n=0}^{\infty} aq^n \qquad |q| < 1$$

converges absolutely because

$$\sum_{n=1}^{\infty} |aq^n|$$

converges

Examples

Example 9

The series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges absolutely for all x

We proved in **Example 1** that it converges for $c > 0$, i.e $c = |x|$

We prove by other methods that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

Examples

Example 10

The Enharmonic series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$$

converges conditionally

True, because we proved that it **converges** and

$$|(-1)^{n+1}\frac{1}{n}| = \frac{1}{n} = |a_n|$$

and so

$$\sum_{n=1}^{\infty}|a_n|$$

diverges

Finite and Infinite Commutativity

Finite Commutativity

We know that finite summation is commutative, i.e.

We have that

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a_{ik}$$

where

a_{ik} is any permutation of $a_1 \ldots a_n$

Infinite Commutativity

The Commutativity fails in the infinite case

For some infinite sums as we showed for example evaluating the infinite sum

$$\sum_{k\geq 0}(-1)^k = \sum_{k=0}^{\infty}(-1)^k = 1-1+1-1+1........$$

in two ways (permutation)

Infinite Commutativity

By grouping (permutating) the sum factors in two different ways:

1. $\sum_{k=0}^{\infty} (-1)^k = (1\text{-}1)+(1\text{-}1)+\ldots = 0$
2. $\sum_{k=1}^{\infty} (-1)^k = 1\text{-}(1\text{-}1)\text{-}(1\text{-}1)\ \ldots = 1$

Question: When and for which **infinite sums** commutativity holds and for which it fails

Infinite Commutativity

Let a_n be a sequence, a_{m_k} is a sequence of permutations of a_n

Definition

A permutation of a set A is any function

$$f : A \xrightarrow[\text{onto}]{\text{1-1}} A \ , \ \textit{where A has any cardinality}$$

In particular

$$f : N \xrightarrow[\text{onto}]{\text{1-1}} N$$

is a permutation of natural numbers and we denote

$$f(n) = m_n$$

Infinite Commutativity

Given an infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ \ldots$$

The infinite series

$$\sum_{k=1}^{\infty} a_{m_k} = a_{m_1} + a_{m_2} + \ \ldots$$

is called its permutation

Infinite Commutativity Theorem

Theorem 11

Every absolutely convergent infinite sum is **commutative**, i.e.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{m_n}$$

for any permutation

$$m_1, \; m_2, \; \dots \; m_n \dots$$

of natural numbers

Infinite Commutativity Theorem

Theorem 11 is **NOT TRUE** for any convergent sum

We can get from a convergent ANHARMONIC series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$

permutations that **converges** or **diverges** to ∞

Riemann Theorem

Theorem 12 Riemann Theorem

For any conditionally convergent infinite sum,
we can transform it by permutation of its factors
into a sum that **diverges** or
to a sum that **converges** to any limit (finite or infinite).