Reminder

$$\boxed{\sum_{k} a_k [P(k)] = \sum_{k \in K} a_k = \sum_{k} a_k [k \in K]}$$

where $K = \{k : P(k)\}$

IN PARTICULAR : when $a_k = 1$, all $k \in K$

$$\boxed{\sum_{k} [P(k)] = \sum_{k \in K} 1}$$

$$\boxed{[P(k)] = \begin{cases} 1 & P(k) \text{ is True} \\ 0 & P(k) \text{ is False} \end{cases}}$$

characteristic function of the predicate $P(k)$

CHARACTERISTIC FUNCTION PROPERTIES

① $[P(n) \wedge Q(n)] = [P(n)] \cdot [Q(n)]$ EXERCISE: prove it.

② $[P(n) \vee Q(n)] = [P(n)] + [Q(n)] - [P(n) \wedge Q(n)]$

$[P(n) \wedge Q(n)] = [P(n)][Q(n)]$

We use ② for summation (particular case)

③ $$\sum_{k} [P(k) \vee Q(k)] = \sum_{k} [P(k)] + \sum_{k} [Q(k)] - \sum_{k} [P \wedge Q(k)]$$

THIS is a PARTICULAR CASE OF

$$\boxed{\sum_{k \in K \cup K'} a_k = \sum_{k \in K} a_k + \sum_{k \in K'} a_k - \sum_{k \in K \cap K'} a_k}$$ where

Take

$p(n) \doteq \lfloor \sqrt[3]{n} \rfloor \mid n$

def

$$\boxed{\lfloor \sqrt[3]{n} \rfloor \mid n \equiv k = \lfloor \sqrt[3]{n} \rfloor \wedge (k \mid n) \equiv k = \lfloor \sqrt[3]{n} \rfloor \wedge (k = n \cdot m}$$

$$\sum_{1 \le n \le 100} \lfloor \sqrt[3]{n} \rfloor \mid n = \sum_{k,n} \left[k = \lfloor \sqrt[3]{n} \rfloor \wedge (k \mid n) \right] [1 \le n \le 1000]$$

$$= \sum_{k,n,m} [k = \lfloor \sqrt[3]{n} \rfloor] [n = km] [1 \le n \le 1000]$$

use $\lfloor x \rfloor = n$ iff $n \le x < n+1$ to $k = \lfloor \sqrt[3]{n} \rfloor$ we get

$k \le \sqrt[3]{n} < k+1$; $k^3 \le n < (k+1)^3$

$$= \sum_{k,n,m} [k^3 \le n < (k+1)^3] [n = km] [1 \le n \le 1000]$$

$(k^3 \le n < (k+1)^3) \wedge (1 \le n \le 1000) \wedge (n = km)$ we get rid of n: (almost)

$(k+1)^3 = 1000$; $k+1 = 10$; $k = 9$; $1 \le k < 10$ we miss $n = 1000$

$$\boxed{(k^3 \le km < (k+1)^3 \wedge 1 \le k < 10) \cup (km = 1000)}$$

$$= \sum_{k,n,m} \left[(k^3 \le km < (k+1)^3) \wedge 1 \le k < 10) \cup \overset{n}{km} = 1000 \right]$$

$n = 1000$

use 3

$$= \sum_{k,n,m} [k^3 \le km < (k+1)^3] [1 \le k < 10]$$

$$= \sum_{k,m} [k^3 \le km < (k+1)^3] + \underbrace{\sum_{n} [n = 1000]}_{1} - \overbrace{\sum_{k,m,n} [k^3 \le km < (k+1)^3 \wedge (1 \le k < 10) \wedge \overset{n}{km} = 1000]}^{0}$$

CONTRAD

$\sum_{n=1000} 1 = 1$

We got

$$\sum_{1\le n\le 1000} \left[\lfloor \sqrt[3]{n} \rfloor \mid n \right] =$$

$$= 1 + \sum_{k,m} \left[k^3 \le km < (k+1)^3 \right] \left[1 \le k < 10 \right]$$

$$= 1 + \sum_{k,m} \left[k^2 \le m < \frac{(k+1)^3}{k} \right] \left[1 \le k < 10 \right]$$

$k^2 \le m < \frac{(k+1)^3}{k}$ iff $m \in \left[k^2 \,..\, \frac{(k+1)^3}{k} \right)$

clo-open INTERVAL

$$= 1 + \sum_{k,m} \left[m \in \left[k^2 \,..\, \frac{(k+1)^3}{k} \right) \right] \left[1 \le k < 10 \right]$$

→ HOW MANY m? (integers)

$[\alpha \,..\, \beta)$ has $\lceil \beta \rceil - \lceil \alpha \rceil$ integers

$$= 1 + \sum_{k} \left(\left\lceil \frac{(k+1)^3}{k} \right\rceil - \lceil k^2 \rceil \right) (1 \le k < 10)$$

$\frac{(k+1)^3}{k} = \frac{k^3+3k^2+3k+1}{k} = k^2+3k+3 + \frac{1}{k}$

Evaluate

$\lceil k^2+3k+3+\frac{1}{k} \rceil - \lceil k^2 \rceil =$

$= k^2+3k+3+\lceil \frac{1}{k} \rceil - k^2$

$= 3k+4$

$\boxed{\lceil x+n \rceil = \lceil x \rceil + n}$

$$= 1 + \sum_{1\le k<10} (3k+4)$$

$$= 1 + \frac{7\cdot 31}{2} \cdot 9 = 172$$

Missing step in evaluation

$$\boxed{\sum_{k,m} [m \in [k^2 .. (k+1)^3/k)] [1 \le k < 10]}$$

To evaluate it we use the following properties and definitions

DEFINITION

① $$\sum_k a_k [P(k)] = \sum_{P(k)} a_k = \sum_{k \in K} a_k$$

where $K = \{k : P(k)\}$

In particular case when $a_k = 1$, all k we get

PROPERTY

Shorthand

② $$\sum_k [P(k)] = \sum_{P(k)} 1 = \sum_{k \in K} 1 = |K| = |P(k)|$$

number of elements of K

CARDINALITY of K

$|K| = |\{k : P(k)\}| \quad k \in \mathbb{Z}$.

DEFINITION

③ $$\sum_{k,m} a_{k,m} [Q(k)][P(m)] =$$

$$= \sum_{Q(k)} \sum_{P(m)} a_{k,m} = \sum_{P(m)} \sum_{Q(k)} a_{k,m}$$

AS PARTICULAR CASE of (3) for

$a_{k,m} = 1$ for all k, m (plus (2))

we get

(4) $$\boxed{\begin{aligned}\sum_{k,m} [P(m)][Q(k)] &= \sum_{Q(k)} \sum_{P(m)} 1 \\ &= \sum_{Q(k)} |P(m)| = \sum_{k} |P(m)|\,[Q(k)]\end{aligned}}$$

$|P(m)| = |\{m \in \mathbb{Z} : P(m)\}|$

IN OUR CASE : $\boxed{Q(k) : \ 1 \le k < 10}$

$\boxed{P(m) : \ m \in [k^2 .. (k+1)^2/k)}$

$[\alpha, \beta)$ has $\lceil\beta\rceil - \lceil\alpha\rceil$ Integers

$$|P(m)| = \lceil (k+1)^2/k \rceil - \lceil k^2 \rceil$$

Integers!

$$= \lceil k^2 + 3k + 3 + 1/k \rceil - \lceil k^2 \rceil = 3k+4$$

We use (4) to evaluate

$$\sum_{k,m} [m \in [k^2 .. (k+1)^2/k)][1 \le k < 10] = \sum_k (3k+4)[1 \le k < 10]$$

$$= \sum_{1 \le k < 10} (3k+4) = 1 + \frac{7+31}{2} \cdot 9 = 172$$

END

CASINO PROBLEM is just a dressed-up version of a mathematical question:

Q HOW MANY integers n, where $1 \le n \le 1000$, satisfy the property $\lfloor \sqrt[3]{n} \rfloor \mid n$. ?

Generalization:

GQ HOW MANY integers n, where $1 \le n \le N$, satisfy the property $\lfloor \sqrt[3]{n} \rfloor \mid n$?

N - denotes here any NATURAL number ≥ 1000. I keep BOOK NOTATION.

HOMEWORK PROBLEM: write all details of the solution of GQ on P. 75 - 76.

SPECTRUM

For any $\alpha \in \mathbb{R}$ we define a SPECTRUM of α as

$$\boxed{Spec(\alpha) = \{ \lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor \dots \}}$$

For some $\alpha \in \mathbb{R}$, $Spec(\alpha)$ is a MULTISET i.e it CAN CONTAIN repeating elements

Example

$\alpha = \frac{1}{2}$, $\lfloor \alpha \rfloor = 0$, $\lfloor 2\alpha \rfloor = 1$, $\lfloor 3\alpha \rfloor = \lfloor \frac{3}{2} \rfloor = 1$

$\lfloor 4\alpha \rfloor = \lfloor 4 \cdot \frac{1}{2} \rfloor = 2$, $\lfloor 5\alpha \rfloor = \lfloor 5/2 \rfloor = 2$ etc

MULTISET

$$\boxed{Spec(1/2) = \{0, 1, 1, 2, 2, 3, 3, 4, 4, 5 \dots\}}$$

$\sqrt{2} \approx 1.4$

$\alpha = \sqrt{2}$ $\lfloor \alpha \rfloor = 1$, $\lfloor 2\alpha \rfloor = \lfloor 2 \cdot \sqrt{2} \rfloor = \lfloor 2.8.. \rfloor = 2$

$\lfloor 3\alpha \rfloor = \lfloor 3\sqrt{2} \rfloor = \lfloor 4.2.. \rfloor = 4$, $\lfloor 4\alpha \rfloor = \lfloor 5.6 \rfloor = 5$

$$\boxed{Spec(\sqrt{2}) = \{1, 2, 4, 5, 7, 8, 9, 11, 12 \dots\}}$$

SET

$$\boxed{Spec(2+\sqrt{2}) = \{3, 6, 10, 13, 17, 20, \dots\}}$$

Observation

Spec($\sqrt{2}$) and Spec($2+\sqrt{2}$) are SETS and form a PARTITION of Natural numbers ($n \geq 1$).

i.e. $Spec(\sqrt{2}) \cap Spec(2+\sqrt{2}) = \phi$

$Spec(\sqrt{2}) \cup Spec(2+\sqrt{2}) = N$

(both are non-empty)

The proof is not straightforward.

It considers two cases ① Finite FACT (any $n \in N$) ② Generalization of the finite Fact to the set of all N.

① FIRST let's look at certain FINITE subsets of Spec($\sqrt{2}$), Spec($2+\sqrt{2}$).

$A_n = \{m \in N : \boxed{m \in Spec(\sqrt{2})} \wedge m \leq n\}$

$B_n = \{m \in N : \boxed{m \in Spec(2+\sqrt{2})} \wedge m \leq n\}$

Remark: Spec($\sqrt{2}$), Spec($2+\sqrt{2}$) are SETS, they are subsets of N.

Example

$A_8 = \{1, 2, 4, 5, 7, 8\}$, $B_8 = \{3, 6\}$

Observe that

① $A_8 \cup B_8 = \{1 \ldots 8\} = \{m : m \le 8\}$

② $A_8 \cap B_8 = \phi$ AND $|A_8| + |B_8| = 8$ ③

Let's check $n = 15$

$A_{15} = \{1, 2, 4, 5, 7, 8, 9, 11, 12, 14, 15\}$

$B_{15} = \{3, 6, 10, 13\}$ AND $|A_{15}| + |B_{15}| = 15$ ③

We get again

① $A_{15} \cup B_{15} = \{1, \ldots 15\} = \{m : m \le 15\}$

② $A_{15} \cap B_{15} = \phi$

We are going to prove that a combination of these ~~two~~ three property hold for all $n \in N, n \geq 1$.

FINITE FACT ①

Given two sets

$A_n = \{m \in N : m \in Spec(\sqrt{2}) \wedge m \le n\}$

$B_n = \{m \in N : m \in Spec(2+\sqrt{2}) \wedge m \le n\}$

The following conditions hold for all $n \geq 1, n \in N$

① $A_n \cap B_n = \phi$, ④ $\forall n,k \ A_k \cap B_n = \phi$

② $A_n \neq \phi$, $B_n \neq \phi$

③ MOREOVER: $A_n \cup B_n = \{1, \ldots n\}$ iff $|A_n| + |B_n| = n$

The FINITE FACT does not YET PROVES that $A_n \cup B_n = \{1, \dots n\}$ but privides a necessary and sufficient CONDITION for it to hold; i.e

(1) $A_n \cup B_n = \{1, \dots n\}$ iff $|A_n| + |B_n| = n$

NEXT STEP: FINITE FACT (2)

we prove that $|A_n| + |B_n| = n$, for all $n \geq 1$

From F.FACTS (1) + (2) we obtain that the following theorem holds

FINITE THEOREM (PARTITION (1))

For any $n \geq 1$, the sets A_n, B_n form a PARTITION of the finite subset $\{1, \dots n\}$ of N.

NEXT STEP: Extend the FINITE THEOREM to the set N

INFINITE THEOREM

The sets Spec($\sqrt{2}$), Spec($2+\sqrt{2}$) form a PARTITION of N ($n \geq 1$).

The BOOK proves ONLY Finite FACT ② and SAYS that from this ~~[illegible]~~ the infinite theorem follows. NOT SO OBVIOUS!

So - we provide here step by step proofs of all what is needed.

FIRST STEP We prove the following generalization of the FINITE FACT ②

GENERAL FACT ②

Let A, B be two non-empty, disjoint subsets of a set $\{1, \dots n\}$, $n \geq 1$ i.e $A, B \subseteq \{1, \dots n\}$, $A \neq \phi$, $B \neq \phi$, $A \cap B = \phi$.

Then the following condition holds

$A \cup B = \{1, \dots n\}$ iff $|A| + |B| = n$

In particular we take $A = A_n$, $B = B_n$ and get FINITE FACT ② as a particular case. ~~because~~ OBSERVE: $1 \in A_n$ (for all $n \geq 1$), $3 \in B_n$ (all $n \geq 1$)

① $A_n \neq \phi$, $B_n \neq \phi$ for all $n \geq 1$.

To prove that

$A_n \cap B_n = \phi$ for all $n \geq 1$

we prove more general statement

$$\boxed{\text{Spec}(\sqrt{2}) \cap \text{Spec}(2+\sqrt{2}) = \phi} \quad (*)$$

Reminder:

$\text{Spec}(\alpha) = \{ \lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor .. \lfloor k\alpha \rfloor .. \}$

Consider $k \geq 1$ and $\lfloor k(2+\sqrt{2}) \rfloor \in \text{Spec}(2+\sqrt{2})$

$\lfloor n + x \rfloor = n + \lfloor x \rfloor$

$$\boxed{\begin{aligned} \lfloor k(2+\sqrt{2}) \rfloor &= \lfloor 2k + k\sqrt{2} \rfloor \\ &= 2k + \lfloor k\sqrt{2} \rfloor \neq \lfloor k\sqrt{2} \rfloor \end{aligned}}$$

all $k \geq 1$

This ~~[illegible]~~ technique proves that all elements of $\text{Spec}(\sqrt{2})$ and $\text{Spec}(2+\sqrt{2})$ are different and the sets are disjoint

In particular $A_n \cap B_k = \phi$, all $n, k \geq 1$

We proved that A_n, B_n satisfy the conditions of the GENERAL FACT (and cond ①, ② of FINITE FACT ①) hence the condition ③ of the FFACT ① holds as a particular case of the General Fact.

Proof of the GFACT follows.

Let $A, B \neq \emptyset$, $A \cap B = \emptyset$, $A, B \subseteq \{1, \dots n\}$ 28

want to show

$A \cup B = \{1, \dots n\}$ iff $|A| + |B| = n$

$\rightarrow$

Let $A \cup B = \{1, \dots n\}$, $|A \cup B| = n$ and

$|A \cup B| = |A| + |B| - |A \cap B|$, but $A \cap B = \emptyset$

so $|A \cup B| = |A| + |B| = n$.

$\leftarrow$

Now let $|A| + |B| = n$ and $|A \cup B| \neq n$

but $|A \cup B| = |A| + |B|$, so $n \neq n$ CONTRADICTION.

Now we are going to prove

FINITE FACT (2)

$$\boxed{|A_n| + |B_n| = n \text{ for all } n \geq 1, n \in N}$$

We want to be able to COUNT the elements of A_n, B_n i.e develop a general formula for $|A_n|$, $|B_n|$

We do it in a GENERAL CASE of any $\alpha \in R$, and spec(α)

DENOTE

$N(\alpha, n)$ = number of elements in the Spec(α) that are $\leq n$

$Spec(\alpha) = \{ \lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor \ldots \}$

$m \in Spec(\alpha)$ iff $m = \lfloor k\alpha \rfloor \wedge k > 0$

$\alpha \in R,\ m \geq 1,\ n \in N$

$N(\alpha, n) = |\{m : m = \lfloor k\alpha \rfloor \wedge m \leq n \wedge k > 0\}|$

$$\boxed{N(\alpha, n) = |\lfloor k\alpha \rfloor : \lfloor k\alpha \rfloor \leq n \wedge k > 0|}$$

$n, k \in N$

$N(\alpha, n) = | P(k) \wedge Q(k) |$ $\quad P(k) : \lfloor k\alpha \rfloor \leq n$

$Q(k) \quad k > 0$

USE PROPERTY

$$\sum_{P(k) \wedge Q(k)} 1 = | P(k) \wedge Q(k) | = \sum_{k} [P(k)][Q(k)]$$

$$= \sum_{Q(k)} [P(k)]$$

$$N(\alpha, n) = \sum_{k} [\lfloor k\alpha \rfloor \leq n][k < 0]$$

$$= \sum_{k > 0} [\lfloor k\alpha \rfloor \leq n]$$

$m \leq n$ iff $m < n+1$

$$= \sum_{k > 0} [\lfloor k\alpha \rfloor < n+1]$$

$$N(\alpha, n) = \sum_{k>0} \left[\lfloor k\alpha \rfloor < (n+1) \right]$$

$\lfloor x \rfloor < n$ iff $x < n$

$$= \sum_{k>0} \left[k\alpha < n+1 \right]$$

$$= \sum_{k} \left[k < \frac{n+1}{\alpha} \right] \left[k > 0 \right]$$

$$= \sum_{k} \left[0 < k < \frac{n+1}{\alpha} \right]$$

$\sum_{k} [P(k)] = \sum_{P(k)} 1 = |P(k)|$

only integer

$$= \sum_{0 < k < \frac{n+1}{\alpha}} 1 = \left| \left\{ k \in \mathbb{Z} : 0 < k < \frac{n+1}{\alpha} \right\} \right|$$

only integers

$$= \left| \left(0 \,..\, \frac{n+1}{\alpha} \right) \right|$$

# of INTEGERS $|(\alpha, \beta)| = \lceil \beta \rceil - \lfloor \alpha \rfloor - 1$

$$= \left\lceil \frac{n+1}{\alpha} \right\rceil - 0 - 1$$

General FORMULA.

$$\boxed{N(\alpha, n) = \left\lceil \frac{(n+1)}{\alpha} \right\rceil - 1}$$

Apply it for $\alpha = \sqrt{2}$, $\alpha = 2 + \sqrt{2}$

NEXT GOAL: prove that

$$\boxed{N(\sqrt{2}, n) + N(2+\sqrt{2}, n) = n}$$

Evaluation

$$N(\alpha, n) = \left\lceil \frac{n+1}{\alpha} \right\rceil - 1$$

$$N(\sqrt{2}, n) + N(2+\sqrt{2}, n) = \left\lceil \frac{n+1}{\sqrt{2}} \right\rceil - 1 + \left\lceil \frac{n+1}{2+\sqrt{2}} \right\rceil - 1$$

$$= \left\lfloor \frac{n+1}{\sqrt{2}} \right\rfloor + \left\lfloor \frac{n+1}{2+\sqrt{2}} \right\rfloor$$

$\lceil x \rceil - 1 = \lfloor x \rfloor$

$\lfloor x \rfloor = x - \{x\}$

$$= \frac{n+1}{\sqrt{2}} - \left\{ \frac{n+1}{\sqrt{2}} \right\} + \frac{n+1}{2+\sqrt{2}} - \left\{ \frac{n+1}{2+\sqrt{2}} \right\}$$

$$= (n+1)\Big(\underbrace{\frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}}}_{1}\Big) - \left(\left\{ \frac{n+1}{\sqrt{2}} \right\} + \left\{ \frac{n+1}{2+\sqrt{2}} \right\} \right)$$

$$= (n+1) - \Big(\underbrace{\left\{ \frac{n+1}{\sqrt{2}} \right\} + \left\{ \frac{n+1}{2+\sqrt{2}} \right\}}_{1} \Big)$$

$$\frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{2+\sqrt{2}+\sqrt{2}}{\sqrt{2}(2+\sqrt{2})} = \frac{2+2\sqrt{2}}{2\sqrt{2}+2} = 1$$

WANT THIS

$= n$

We are going now to prove that

$$\left\{ \frac{n+1}{\sqrt{2}} \right\} + \left\{ \frac{n+1}{2+\sqrt{2}} \right\} = 1$$

Observe that we proved that

$$\frac{n+1}{\sqrt{2}} + \frac{n+1}{2+\sqrt{2}} = n+1$$

I really have to prove that

IF $\dfrac{n+1}{\sqrt{2}} + \dfrac{n+1}{2+\sqrt{2}} = n+1$

THEN

$$\left\{ \frac{n+1}{\sqrt{2}} \right\} + \left\{ \frac{n+1}{2+\sqrt{2}} \right\} = 1$$

This is enough

We did already proved

$$\frac{n+1}{\sqrt{2}} + \frac{n+1}{2+\sqrt{2}} = n+1$$

We prove a more general fact

FACT

$$\boxed{\begin{array}{l} \text{Given } x_1, x_2 \notin \mathbb{Z} \text{ (non-itegers)} \\ \text{If } \quad x_1 + x_2 = n+1 \\ \text{then } \quad \{x_1\} + \{x_2\} = 1 \qquad n \in \mathbb{Z} \end{array}}$$

[illegible]

[illegible]

In our case $x_1 = \dfrac{n+1}{\sqrt{2}}$, $x_2 = \dfrac{n+1}{2+\sqrt{2}}$

Proof.

$x_1 = \lfloor x_1 \rfloor + \{x_1\}$, $x_2 = \lfloor x_2 \rfloor + \{x_2\}$

we have

$0 \le \{x_1\} < 1$

$x_1 + x_2 = \lfloor x_1 \rfloor + \{x_1\} + \lfloor x_2 \rfloor + \{x_2\} = n+1 \qquad 0 \le \{x_2\} < 1$

$\{x_1\} + \{x_2\} + \underbrace{\lfloor x_1 \rfloor + \lfloor x_2 \rfloor}_{\text{integers}} = n+1$

$\{x_1\} \ne 0$

$\{x_2\} \ne 0$

x_1, x_2 NONINT.

$0 \le \{x_1\} < 1$

$0 \le \{x_2\} < 1$

$0 \le \{x_1\} + \{x_2\} < 2$ so $\boxed{\{x_1\} + \{x_2\} = 1}$

$n+1 = m + \theta$, where $0 < \theta < 2$, so $\theta = 1$

m Int. and $m = n$

We have proved

FINITE THEOREM (PARTITION THEOREM)

The sets A_n, B_n form a partition of the set $\{1, .. n\}$, for all $n \geq 1$, $n \in N$.

NEXT : INFINITE PARTITION THEOREM

$Spec(\sqrt{2})$, $Spec(2+\sqrt{2})$ form a partition of $N - \{0\}$.

Reminder

$A_n = \{m \in Spec(\sqrt{2}) : m \leq n\}$

$B_n = \{m \in Spec(2+\sqrt{2}) : m \leq n\}$

FACTS (about A_n, B_n)

① $\forall_{n \geq 1} (A_n \subseteq A_{n+1} \wedge B_n \subseteq B_{n+1})$

$\{A_n\}$ is monotonically increasing sequence of sets. $\{B_n\}$ the same

② $\forall_{\substack{n \geq 1 \\ k \geq 1}} (A_n \cap B_k = \Phi \wedge A_n \cup B_n = \{1, ... n\})$

③ $Spec(\sqrt{2}) = \bigcup_{n \geq 1} A_n$

$Spec(2+\sqrt{2}) = \bigcup_{n \geq 1} B_n$

① follows directly from the definition ($m \le n+1$); ② was proved already

③ $m \in Spec(\sqrt{2})$ iff $\exists k \geq 1\ m = \lfloor k(\sqrt{2}) \rfloor$ iff $\exists n = k\ m \in A_m$ iff $m \in \bigcup_{n \geq 1} A_n$. Same for B_n.

INFINITE PARTITION THEOREM (re-stated)

For any sets A_m, B_n the following conditions hold

① $\bigcup_{n \geq 1} A_n \neq \emptyset$, $\bigcup_{n \geq 1} B_n \neq \emptyset$

② $\bigcup_{n \geq 1} A_m \cap \bigcup_{n \geq 1} B_n = \emptyset$

③ $\bigcup_{n \geq 1} A_n \cup \bigcup_{n \geq 1} B_n = N - \{0\}$

i.e. the sets $\bigcup A_m = Spec(\sqrt{2})$ and $\bigcup B_n = Spec(2+\sqrt{2})$ form a PARTITION of $N - \{0\}$

① is true as $\forall n (A_n \neq \emptyset \wedge B_n \neq \emptyset)$

② Assume $\bigcup_n A_m \cap \bigcup_n B_n \neq \emptyset$ i.e. there is x, $x \in \bigcup_n A_n \wedge x \in \bigcup_n B_n$ iff $x \in A_k \wedge x \in B_m$ contradiction with $A_k \cap B_m = \emptyset$ all k, m

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③ Assume
$\boxed{x \in \bigcup_{n\geq 1} A_n \cup \bigcup_{n\geq 1} B_n \text{, show } x \in N - \{0\}}$

$x \in \bigcup_{n\geq 1} A_n \vee x \in \bigcup_{n\geq 1} B_n$ iff $\exists_{k\geq 1}\, x \in A_k$

$\wedge\ \exists_{m\geq 1}\, x \in B_m$. $\boxed{\text{Cases: } n=k,\ n>k,\ n<k.}$

$x \in A_n \vee x \in B_n$ iff $x \in (A_n \cup B_n)$

n=k. We get $x \in A_n \vee x \in B_n$ so $x \in \{1, \ldots n\} \subseteq N - \{0\}$

and $A_n \cup B_n = \{1, \ldots n\}$

and $x \in N - \{0\}$.

n>k $x \in A_m \wedge x \in B_k$. But by FACT ①

$\{B_n\}$ is increasing, so $B_k \subseteq B_n$ for $n > k$

and $x \in B_n$. So $x \in A_n \cup B_n = \{1 \ldots n\}$ and

$x \in N - \{0\}$.

n<k $x \in A_n \wedge x \in B_k$. But $\{A_m\}$ is increasing

so $A_n \subseteq A_k$ for $k > n$; $x \in A_k$ and

$x \in A_n \cup B_k = \{1, \ldots k\} \subseteq N - \{0\}$ and $x \in N - \{0\}$.

$\boxed{\text{Assume } x \in N - \{0\} \text{, show } x \in \bigcup_{n\geq 0} A_n \cup \bigcup_{n\geq 0} B_n}$

Proof by contradiction.

Let $x \in N - \{0\}$ and $x \notin \bigcup A_n \cup \bigcup B_n$ iff

$x \notin \bigcup_{n\geq 1} A_n \wedge x \notin \bigcup_{n\geq 1} B_n$ iff $\forall_k\, x \notin A_k \wedge \forall_m\, x \notin B_m$

But $A_n \cup B_n = \{1, \ldots n\}$, so $x \in A_n \vee x \in B_n$

there is an $x \in \{1, \ldots n\}$ CONTRADICTION

FLOOR/CEILING SUMS

EXAMPLE: Evaluate

$$\sum_{0\le k<n} \lfloor\sqrt{k}\rfloor$$

Observation: $\displaystyle\lfloor\sqrt{k}\rfloor_{k\geq 0} = \sum_{\substack{m=\lfloor\sqrt{k}\rfloor \\ m\geq 0}} m$

$$\sum_{0\le k<n} \lfloor\sqrt{k}\rfloor = \sum_{0\le k<n} \sum_{\substack{m\geq 0 \\ m=\lfloor\sqrt{k}\rfloor}} m$$

$$= \sum_{0\le k<n} \sum_{m\geq 0} m\left[m=\lfloor\sqrt{k}\rfloor\right] = \sum_{m\geq 0}\sum_{0\le k<n} m\left[m=\lfloor\sqrt{k}\rfloor\right]$$

$$= \sum_{m\geq 0}\sum_{k\geq 0} m\left[m=\lfloor\sqrt{k}\rfloor\right][k<n]$$

$\lfloor x\rfloor = n$ iff $n\le x<n+1$

$$= \sum_{m,k\geq 0} m[k<n]\cdot[m\le\sqrt{k}<m+1]$$

$$= \sum_{m,k\geq 0} m[k<n \wedge m^2\le k<(m+1)^2]$$

$$\boxed{\sum_{0\le k<n} \lfloor\sqrt{k}\rfloor = \sum_{m,k\ge 0} m\left[m^2 \le k < (m+1)^2 \wedge k<n\right]}$$

Let's look at $P(k,m,n)$: $m^2 \le k < (m+1)^2 \wedge k<n$

$P(k,m,n) \equiv m^2 \le k < n < (m+1)^2 \vee m^2 \le k < (m+1)^2 \le n$

$P(k,m,n) = Q \vee R$

$$\sum_{m,k}[Q\vee R] = \sum_{m,k} Q + \sum_{m,k} R - \sum_{m,k} Q\wedge R$$

$Q\wedge R$ is False, so $\sum Q\wedge R = 0$ and we get

$$\boxed{\sum_{0\le k<n}\lfloor\sqrt{k}\rfloor = \text{①}\sum_{m,k\ge 0} m\left[m^2\le k<(m+1)^2\le n\right] + \text{②}\sum_{m,k\ge 0} m\left[m^2 \le k < n < (m+1)^2\right]}$$

ASSUME that $\boxed{n = a^2 \text{ for } a\in N}$

Examine ②

$m^2 \le k < a^2 < (m+1)^2$

is a FALSE statement, there is no $a\in N$

$m \le a < (m+1)$

$m^2 \le a^2 < (m+1)^2$

so second sum ② is $= 0$

$$\sum_{0\le k<n} \lfloor\sqrt{k}\rfloor = \sum_{k,m\ge 0} m\left[m^2\le k<(m+1)^2\le a^2\right]$$

$$m^2\le k<(m+1)^2\le a^2 \equiv m^2\le k<(m+1)^2 \wedge$$
$$\wedge (m+1)^2\le a^2 \equiv m^2\le k<(m+1)^2 \wedge m+1\le a$$

$$= \sum_{k,m\ge 0} m\left[m^2\le k<(m+1)^2\right]\left[m+1\le a\right]$$

$$= \sum_{m\ge 0}\sum_{k\ge 0} \underbrace{m[m+1\le a]}_{\text{wo } k}\left[m^2\le k<(m+1)^2\right]$$

$$= \sum_{m\ge 0} m[m+1\le a]\sum_{k\ge 0}\left[m^2\le k<(m+1)^2\right]$$

$\sum [P(k)] = \sum_{P(k)} 1 = |P(k)|$

$$= \sum_{m\ge 0} m[m+1\le a]\sum_{k\ge 0}\left[\left[m^2 .. (m+1)^2\right)\right]$$

$|[\alpha..\beta)|_{\text{int}} = \lceil\beta\rceil-\lceil\alpha\rceil$

$$= \sum_{m\ge 0} m(2m+1)[m+1\le a]$$

$(m+1)^2-m^2 = 2m+1$

$$= \sum_{m\ge 0}(2m^2+m)[m+1\le a] = \sum_{m\ge 0}\left(2m^{\underline{2}}+3m^{\underline{1}}\right)[m+1\le a]$$

$2m^2+m = 2m^2-2m+2m+m$
$= \underbrace{2m(m-1)+3m} = 2m^{\underline{2}}+3m^{\underline{1}}$

$x^{\underline{2}} = x(x-1) = x^2-x$
$x^{\underline{1}} = x$

$$\sum_{0\le k<n} \lfloor\sqrt{k}\rfloor = \sum_{m\ge 0}\left(2m^{\underline{2}} + 3m^{\underline{1}}\right)[m+1\le a]$$

$m+1\le a$

$\equiv$

$m<a$

$$= \sum_{m\ge 0}\left(2m^{\underline{2}} + 3m^{\underline{1}}\right)[m<a]$$

$$= \sum_{0\le m<a}\left(2m^{\underline{2}} + 3m^{\underline{1}}\right)$$

$$= \sum\nolimits_0^a \left(2m^{\underline{2}} + 3m^{\underline{1}}\right)\delta m$$

$$= 2\frac{m^{\underline{3}}}{3} + 3\frac{m^{\underline{2}}}{2}\Big|_0^a$$

$$= \frac{2}{3}m(m-1)(m-2) + \frac{3}{2}m(m-1)\Big|_0^a$$

$$= \frac{2}{3}a(a-1)(a-2) + \frac{3}{2}a(a-1)$$

$$= a(a-1)\left(\frac{2}{3}(a-2) + \frac{3}{2}\right)$$

$\frac{2}{3}a - \frac{2}{3} + \frac{3}{2} =$

$= \frac{4a}{6} + \frac{1}{6} = \frac{1}{6}(4a+1)$

$$\boxed{\sum_{0\le k<n}\lfloor\sqrt{k}\rfloor = \frac{1}{6}(a-1)a(a+1) \qquad n=a^2}$$

end.

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DO CASE

$$\boxed{a = \lfloor\sqrt{n}\rfloor}$$