Chapter 3
INTEGER FUNCTIONS

FLOOR For any $x \in R$ (real) we define:

$\lfloor x \rfloor$ = the greatest integer less than or equal x

CEILING

$\lceil x \rceil$ = the least integer greater than or equal to x

SYMBOLIC

FLOOR

$$\lfloor x \rfloor = \max\{a \in Z : a \le x\}$$

unique max = greatest

CEILING

$$\lceil x \rceil = \min\{a \in Z : a \ge x\}$$

unique min = the least

$P_1 = (\{a \in Z : a \le x\}, \le)$ P_1 has unique max " greatest i.e $\lfloor x \rfloor$ exists

$P_2 (\{a \in Z : a \geq x\}, \le)$ P_2 has unique min " least $\lceil x \rceil$ exists

FACT $\boxed{\text{For any } x \in \mathbb{R},\ \lfloor x \rfloor, \lceil x \rceil \text{ exist and are unique}}$

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We can hence define functions

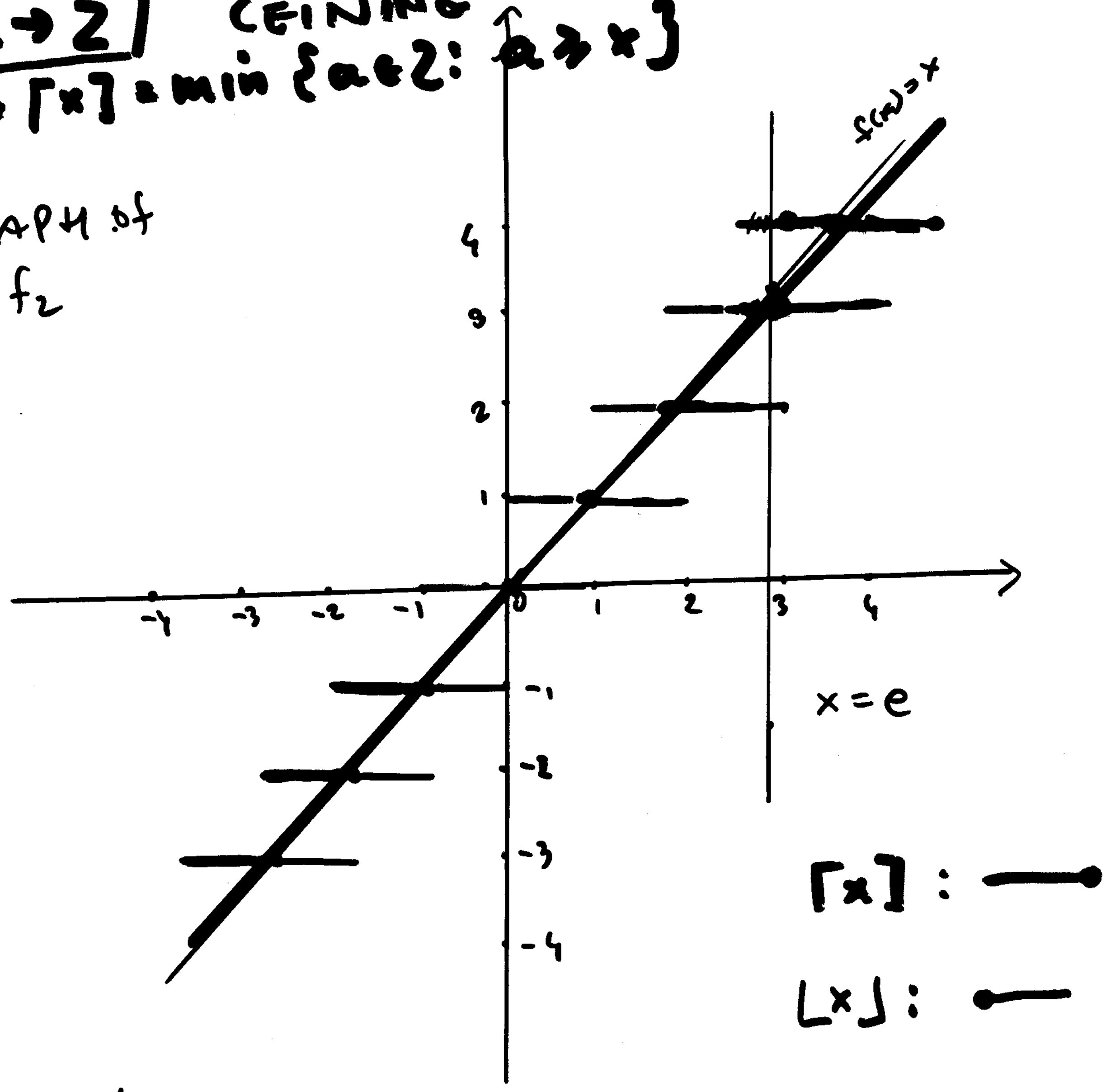
$\boxed{f_1 : \mathbb{R} \to \mathbb{Z}}$ FLOOR

$f_1(x) = \lfloor x \rfloor = \max\{a \in \mathbb{Z} : a \leq x\}$ and

$\boxed{f_2 : \mathbb{R} \to \mathbb{Z}}$ CEILING

$f_2(x) = \lceil x \rceil = \min\{a \in \mathbb{Z} : a \geq x\}$

GRAPH of f_1, f_2



We read

$\lfloor e \rfloor = 2 \qquad \lceil e \rceil = 3$

PROPERTIES of $\lfloor x \rfloor$, $\lceil x \rceil$

① $\lfloor x \rfloor = x$ iff $x \in \mathbb{Z}$

$\lceil x \rceil = x$ iff $x \in \mathbb{Z}$

② $x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$

③ $\lfloor -x \rfloor = -\lceil x \rceil$, $\lceil -x \rceil = -\lfloor x \rfloor$

④ $\lceil x \rceil - \lfloor x \rfloor = \begin{cases} 0 & x \in \mathbb{Z} \\ 1 & x \notin \mathbb{Z} \end{cases} = [x \notin \mathbb{Z}]$

characteristic function notation

MORE PROPERTIES

$x \in \mathbb{R}$, $n \in \mathbb{Z}$

5. $\lfloor x \rfloor = n$ iff $n \le x < n+1$

Directly from definition

6. $\lfloor x \rfloor = n$ iff $x - 1 < n \le x$

TRUE!

$\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$

7. $\lceil x \rceil = n$ iff $n - 1 < x \le n$

8. $\lceil x \rceil = n$ iff $x \le n < x + 1$

Apply iff above

$\lfloor x \rfloor + n \le x + n \le \lfloor x \rfloor + n + 1$

9. $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ iff (by 5) $n \in \mathbb{Z}$

But $\lfloor n x \rfloor \ne n \lfloor x \rfloor$

$n = 2$, $x = \frac{1}{2}$

$\lfloor 2 \cdot \frac{1}{2} \rfloor = 1 \ne 2 \lfloor \frac{1}{2} \rfloor = 0$

$\boxed{\text{MORE PROPERTIES}}$ $x \in \mathbb{R}$, $n \in \mathbb{Z}$ (Insert $\lfloor x \rfloor$, $\lceil x$

(10) $x < n$ iff $\lfloor x \rfloor < n$

(11) $n < x$ iff $n < \lceil x \rceil$

(12) $x \leq n$ iff $\lceil x \rceil \leq n$

(13) $n \leq x$ iff $n \leq \lfloor x \rfloor$

Proof of 10

$\rightarrow$ let $x < n$, so $\lfloor x \rfloor < n$ as $\lfloor x \rfloor \leq x$

$\leftarrow$ let $\lfloor x \rfloor < n$ by (2) $x - 1 < \lfloor x \rfloor$ i.e $x < \lfloor x \rfloor + 1$

by $\lfloor x \rfloor < n$ we get $\lfloor x \rfloor + 1 \leq n$ so we get $x < n$.

$x < \lfloor x \rfloor + 1 \leq n$ and

FACTORIAL PART of x; $\{x\}$ ~~Another~~ !

$$\boxed{\{x\} = x - \lfloor x \rfloor}$$

I~~NTEGER~~

$0 \leq \{x\} < 1$

Write ~~[illegible]~~ $x = \{x\} + \lfloor x \rfloor$

$x = \lfloor x \rfloor + \{x\}$

FACT!

A $x = n + \theta$, $n \in \mathbb{Z}$ Integer and $0 \leq \theta < 1$

then $n = \lfloor x \rfloor$ and $\theta = \{x\}$

(5) $\lfloor x \rfloor = n$ iff $n \leq x < n+1$

$x = n + \theta$ then $n \leq x < n+1$ iff $\lfloor x \rfloor = n$

we get

$x = \lfloor x \rfloor + \theta$ so $\theta = \{x\}$

We proved

(1)

$$\boxed{\lfloor x+n \rfloor = \lfloor x \rfloor + n \quad , \quad n \in \mathbb{Z}, \; x \in R}$$

Question

WHAT happens when we consider

$\lfloor x + y \rfloor$, $x \in R$, $y \in R$

$0 \le \{x\} < 1$

$0 \le \{y\} < 1$

Let's look.

$x = \lfloor x \rfloor + \{x\}$, $y = \lfloor y \rfloor + \{y\}$

used $\lfloor x+n \rfloor = \lfloor x \rfloor + n$

$$\lfloor x+y \rfloor = \lfloor \lfloor x \rfloor + \lfloor y \rfloor + \{x\} + \{y\} \rfloor$$

$$\overset{(1)}{=} \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \{x\} + \{y\} \rfloor$$

and $0 \le \{x\} + \{y\} < 2$ so we get

FACT

$$\lfloor x+y \rfloor = \begin{cases} \lfloor x \rfloor + \lfloor y \rfloor & \text{when } 0 \le \{x\} + \{y\} < 1 \\ \lfloor x \rfloor + \lfloor y \rfloor + 1 & \text{when } 1 \le \{x\} + \{y\} < 2 \end{cases}$$

EXAMPLE

① $\boxed{\text{FIND } \lceil \log_2 35 \rceil}$

Observe

$$2^5 < 35 \leq 2^6$$

$$\log_2 2^5 < \log_2 35 \leq \log_2 2^6$$

$$\boxed{5 < \log_2 35 \leq 6}$$

⑦ $\lceil x \rceil = n$ iff $n-1 < x \leq n$

We get

$$\lceil \log_2 35 \rceil = 6$$

② $\boxed{\text{FIND } \lceil \log_2 32 \rceil}$

$$2^4 < 32 \leq 2^5$$

by ⑦ we get

$$4 < \log_2 32 \leq 5$$

$$\lceil \log_2 32 \rceil = 5$$

EXAMPLE

$\boxed{\text{FIND } \lfloor \log_2 35 \rfloor, \lfloor \log_2 32 \rfloor}$

Observe

$2^5 \leq 35 < 2^6$

$5 \leq \log_2 35 < 6$

$\boxed{\lfloor \log_2 35 \rfloor = 5}$

5

$\boxed{\lfloor x \rfloor = n \text{ iff } n \leq x < n+1}$

$\log_2 32 = 5$

$\boxed{\lfloor \log_2 32 \rfloor = 5} = \lceil \log_2 32 \rceil$

OBSERVE: $35 = (100011)_2$

35 has 6 digits in binary expansion and $\lceil \log_2 35 \rceil = 6$

$\lfloor \log_2 35 \rfloor = 5$

QUESTION: is it TRUE/FALSE ? NO!

$\boxed{\#\text{digits of b.exp of } n = \lceil \log_2 n \rceil}$

$32 = (100000)$

and $\lceil \log_2 32 \rceil = 5 \neq 6$

QUESTION:

Can we develop a connection (formula) between

$\lfloor \log_2 n \rfloor$ and # of digits (m) in the binary representation of n? ($n > 0$)

YES Let $n \neq 0$, $n \in \mathbb{N}$ such that n has m BITS in binary representation. Hence we have

$$2^{m-1} \leq n < 2^m \qquad n = \underbrace{a_{m-1} 2^{m-1} + \dots + a_1 + a_0}_{m\text{-digits}}$$

$$m-1 \leq \log_2 n < m$$

iff

$$\lfloor \log_2 n \rfloor = m-1$$

and

$$\boxed{m = \lfloor \log_2 n \rfloor + 1}$$

Exercise

DO THE SAME FOR $\lceil \log_2 n \rceil$

$n = 35$

$m = \lfloor \log_2 35 \rfloor + 1 = 5+1 = 6$

$n = 32$

$m = \lfloor \log_2 32 \rfloor + 1 = 5+1 = 6$

EXERCISE

PROVE that

$$\forall (x \in R \wedge x \geq 0) \quad \lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$$

i.e

$$\forall x \left(x \in R \wedge x \geq 0 \Rightarrow \lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor \right)$$

or just simply

FACT 2

$$\boxed{\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor \quad \text{for all } x \in R,\ x \geq 0}$$

Proof Take $\lfloor \sqrt{\lfloor x \rfloor} \rfloor$. First we get rid of outside $\lfloor\ \rfloor$ + of $\sqrt{n}$ and then of $\lfloor x \rfloor$

LET

$\boxed{m = \lfloor \sqrt{\lfloor x \rfloor} \rfloor}$ iff ⑤ $\lfloor x \rfloor = m$ iff $m \leq x < m+1$

$m \leq \sqrt{\lfloor x \rfloor} < m+1$

use: ⑬ $n \leq x$ iff $n \leq \lfloor x \rfloor$

get: $m^2 \leq x$

$m^2 \leq \lfloor x \rfloor < (m+1)^2$

use: ⑩ $\lfloor x \rfloor < n$ iff $x < n$

$x \leq (m+1)^2$

$\boxed{m^2 \leq x < (m+1)^2}$ STOP

LET $\lfloor \sqrt{x} \rfloor = n$ iff

$n \leq \sqrt{x} \leq n+1$

$n^2 \leq x \leq (n+1)^2$

iff $n = m$ and

$$\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$$

end.

we prove in a similar way (exercise!)

FACT 3

$$\boxed{\lceil \sqrt{\lceil x \rceil} \rceil = \lceil \sqrt{x} \rceil \quad \text{for all } x \in R,\ x \geq 0}$$

QUESTION: $\sqrt{x}$ is a particular $f: R^{+} \to R$
$f(x) = \sqrt{x}$

CAN we have a SIMILAR property for other functions $f: R \to R$ (WHICH?)

ANSWER: YES. when f is monotonic and continuous and Increasing: i.e we will prove:

FACT 4

Let $f: R' \to R$ (maybe $R' = R$, $R' = R^{+}$ etc.)
$f = f(x)$ be such that f is CONTINOUS, MONOTONIC and INCREASING on its domain R'.

If additionally f has the following property

(P) $\boxed{\text{If } f(x) \in Z, \text{ then } x \in Z}$

then

$$\lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor \quad \text{and}$$

$$\lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil \quad \text{for all}$$

$x \in R'$ for which (P) holds

proof $\boxed{\lceil f(\lceil x \rceil) \rceil = \lceil f(x) \rceil}$ under the assumptions (P) + monot + increasing + continuous

① $x = \lceil x \rceil$ we get $x \in \mathbb{Z}$ skip $\lceil\,\rceil$

$\lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil$ trivial

② $x \neq \lceil x \rceil$. By definition $x < \lceil x \rceil$ and by monotonic fun $f(x) \leq f(\lceil x \rceil)$, and by non-decreasing $\lceil\,\rceil$ ($x < y$ then $\lceil x \rceil \leq \lceil y \rceil$) we get $\boxed{\lceil f(x) \rceil \leq \lceil f(\lceil x \rceil) \rceil.}$ Now we show that "$<$" is impossible – hence we will have "$=$". Assume $\lceil f(x) \rceil < \lceil f(\lceil x \rceil) \rceil$.

By $x < \lceil x \rceil$ we get

$$f(x) < \lceil f(x) \rceil < \lceil f(\lceil x \rceil) \rceil$$

f is continuous, then there is y, such that

$$f(y) = \lceil f(x) \rceil \quad \text{and}$$

$$f(x) < f(y) < f(\lceil x \rceil) \qquad \text{f cont and monot increasing}$$

so this holds when

$$x < y < \lceil x \rceil \qquad \text{But } x \leq \lceil x \rceil$$

so we get $\boxed{x \leq y < \lceil x \rceil}$ ① (there is such y!)

But $f(y) = \lceil f(x) \rceil$ i.e $f(y) \in \mathbb{Z}$, hence by (P), we get : $\boxed{y \in \mathbb{Z}}$ ② There is no $y \in \mathbb{Z}$.

①+② are contradictory ! $x \leq y < \lceil x \rceil$ QED

Special CASE of FACT 4 (for $\lfloor\ \rfloor$)

$$\boxed{\begin{aligned} \left\lfloor \frac{x+m}{n} \right\rfloor &= \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor \\ \left\lceil \frac{x+m}{n} \right\rceil &= \left\lceil \frac{\lceil x \rceil + m}{n} \right\rceil \end{aligned}}$$

$f(x) = \frac{x+m}{n}$

$n, m \in \mathbb{Z}$
$n > 0$

FACT 5

$\boxed{f(x) = \frac{x}{n} + \frac{m}{n}}$

like with any positive a
$y = ax + b$

Example
Take $m = 0$, $n = 10$
Evaluate

$$\left\lfloor \left\lfloor \left\lfloor \frac{x}{10} \right\rfloor / 10 \right\rfloor / 10 \right\rfloor = \left\lfloor \left\lfloor {}^{x}/_{10}/_{10} \right\rfloor / 10 \right\rfloor$$

$$= \left\lfloor \left\lfloor \frac{x}{100} \right\rfloor / 10 \right\rfloor = \left\lfloor \frac{x}{1000} \right\rfloor$$

Dividing x three times by 10 and throwing off digits is the same as dividing x by 1000 and throwing out the remainder.

Integers in the INTERVALS

Interval (closed)

$[\alpha, \beta] = \boxed{\{x \in R : \alpha \le x \le \beta\}}$

STANDARD NOTATION

$= [\alpha \,..\, \beta]$ ← BOOK NOTATION

We use book notation, because $[P(x)]$ denotes (in the book) the characteristic function

$(\alpha, \beta) = \boxed{\{x \in R : \alpha < x < \beta\} = (\alpha \,..\, \beta)}$

OPEN INTERVAL

$[\alpha, \beta) = \{x \in R : \alpha \le x < \beta\} = [\alpha \,..\, \beta)$

$(\alpha, \beta] = \{x \in R : \alpha < x \le \beta\} = (\alpha \,..\, \beta]$

HALF OPEN INTERVALS

PROBLEM :

$A = \{n \in Z : \alpha \le n \le \beta\}$

FIND $|A|$

$<$	$<$
$\le$	$<$
$<$	$\le$

HOW MANY ARE there INTEGERS IN THE INTERVALS OF REAL NUMBERS

WE BRING BACK OUR $\lceil\ \rceil$, $\lfloor\ \rfloor$ PROPERTIES

$$\boxed{\begin{array}{lll} \alpha \le n < \beta & \text{iff} & \lceil\alpha\rceil \le n < \lceil\beta\rceil \\ \alpha < n \le \beta & \text{iff} & \lfloor\alpha\rfloor < n \le \lfloor\beta\rfloor \end{array}}$$

$[\alpha \dots \beta)$ contains exactly $\lceil\beta\rceil - \lceil\alpha\rceil$ integers

$(\alpha \dots \beta]$ contains $\lfloor\beta\rfloor - \lfloor\alpha\rfloor$ integers

$[\alpha \dots \beta]$ contains $\lfloor\beta\rfloor - \lceil\alpha\rceil + 1$ int.

and we must assume $\alpha \ne \beta$ to evaluate:

$(\alpha \dots \beta)$ contains $\lceil\beta\rceil - \lfloor\alpha\rfloor - 1$

(because $(\alpha \dots \alpha) = \emptyset$ and can't contain -1 int)

INTERVAL	# INTEGERS	RESTRICTIONS
$[\alpha \dots \beta]$	$\lfloor\beta\rfloor - \lceil\alpha\rceil + 1$	$\alpha \le \beta$
$[\alpha \dots \beta)$	$\lceil\beta\rceil - \lceil\alpha\rceil$	$\alpha \le \beta$
$(\alpha \dots \beta]$	$\lfloor\beta\rfloor - \lfloor\alpha\rfloor$	$\alpha \le \beta$
$(\alpha \dots \beta)$	$\lceil\beta\rceil - \lfloor\alpha\rfloor - 1$	$\alpha < \beta$

CASINO PROBLEM

There is a roulette wheel with 1,000 slots (numbered 1 ... to 1,000)

IF the number n that comes up on a spin is divisible by $\lfloor \sqrt[3]{n} \rfloor$ i.e

$$\lfloor \sqrt[3]{n} \rfloor \mid n$$

THEN n is a WINNER.

In the game cassino pays \$5 if you are the winner; but the looser has to pay \$1.

Q CAN we expect to make money if we play this game?

Let's compute AVERAGE winnings ie amount we win (or lose) per play

W - # of winners

$L = 1000 - W$ # of loosers

If Each number comes once during 1000 plays, we win $5W$ and lose L dollars

AVERAGE WINNINGS in 1000 plays

$$\frac{5W - L}{1000} = \frac{5W - (1000 - W)}{1000} = \frac{6W - 1000}{1000}$$

We have advantage if

$\frac{6W - 1000}{1000} > 0$, $6W > 1000$, $\boxed{W > 167}$

$\boxed{\text{ANSWER}}$: If there is 167 or more winners (and each number comes up only once) then we have the advantage, otherwise the CASINO wins.

PROBLEM:

$\boxed{\text{How to count the number of WINNERS among 1 to 1000}}$

METHOD:

Use summation

characteristic function

$$\boxed{W = \sum_{n=1}^{1000} [n \text{ is a winner}]}$$

17
17a

BOOK SOLUTION   QUESTIONS ① what does $\sqrt[3]{\ }$ mean?

① $W = \sum_{n=1}^{1000} [n \text{ is a winner}] = \sum_{1 \le n \le 1000} [\lfloor \sqrt[3]{n} \rfloor \mid n]$

② why ① = ② + ③ what is $[k = \lfloor\sqrt[3]{n}\rfloor]\cdot[k \mid n]\cdot[\ \ ]$?

② $= \sum_{k,n} [k = \lfloor \sqrt[3]{n} \rfloor][k \mid n][1 \le n \le 1000]$

explain equivalence

③ $= \sum_{k,m,n} [k^3 \le n < (k+1)^3][n = km][1 \le n \le 1000]$

TRANSITION

④ $= 1 + \sum_{k,m} [k^3 \le km < (k+1)^3][1 \le k < 10]$

How we get $1 + \Sigma$?   How we change $\sum_{k,m,n}$ to $\sum_{k,m}$?

⑤ $= 1 + \sum_{k,m} [m \in [k^2 \ldots (k+1)^3/k)][1 \le k < 10]$

what is this! explain TRANSITION!

⑥ $= 1 + \sum_{1 \le k < 10} (\lceil k^2 + 3k + 3 + 1/k \rceil - \lceil k^2 \rceil)$

check

⑦ $= 1 + \sum_{1 \le k < 10} (3k + 4) = 1 + \frac{7+31}{2} \cdot 9 = 172$

BOOK COMMENT: the only "difficult" maneuver is the decision between lines ③ and ④ to treat $n = 1000$ as a special case.
(The ineq. $k^3 \le n < (k+1)^3$ does not combine easily with $1 \le n \le 1000$ when $k = 10$)