CHAPTER 5
PROBLEMS 4 &6

Problem 4: Evaluate $\binom{-1}{k}$ by negating (actually un-negating) its upper index

We know that Binomial coefficient

$$\binom{r}{k} = \begin{cases} \dfrac{r(r-1)\ldots(r-k+1)}{k(k-1)\ldots(1)} = \dfrac{r^{\underline{k}}}{k!}, & \text{integer } k \geq 0 \\ 0, & \text{integer } k < 0 \end{cases}$$

where
r is a real number ,k is an integer

The above definition can be recast in terms of factorials in the common case that the upper index r is an positive integer n (natural number), that's greater than or equal to the lower index k:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad \text{integers } n \geq k \geq 0$$

Now to get the values of the coefficients for different values of n and k, we have the Pascal's Triangle

n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$	$\binom{n}{7}$	$\binom{n}{8}$	$\binom{n}{9}$	$\binom{n}{10}$
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

Observation: Every number in Pascal's triangle is the sum of two numbers in the previous row, one directly above it and the other just to the left.

So this gives the addition formula

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$$ Text book definition (5.8)

Where r is a positive integer and k is an integer

Now let's look more closely at the values of when n is a negative integer. One way to approach these values is to use the addition law and obtain the below table. For example, we must have $\binom{-1}{0} = 1$, **since** $\binom{0}{0} = \binom{-1}{0} + \binom{-1}{-1}$ **and**

$\binom{-1}{-1} = 0$; then we must have $\binom{-1}{0} = -1$,

And $\binom{0}{1} = \binom{-1}{1} + \binom{-1}{0}$; **and so on.**

Pascal 's Triangle extended upwards

n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$	$\binom{n}{7}$	$\binom{n}{8}$	$\binom{n}{9}$	
-4	1	-4	10	-20	35	-56	84	-120	165	-220	286
-3	1	-3	6	-10	15	-21	28	-36	45	-55	66
-2	1	-2	3	-4	5	-6	7	-8	9	-10	11
-1	1	-1	1	-1	1	-1	1	-1	1	-1	1
0	1										

For negative n we can expand the binomial coefficient as

$r^{\underline{k}} = r\,(r-1)\ldots(r-k+1)$ (here we take r to get a general case)

$= (-1)^k\,(-r)\,(1-r)\ldots(k-1-r)$ (we have -1 common,as all factors are negative)

$= (-1)^k\,(k-r-1)^{\underline{k}}$

when k ≥ 0, and when k < 0 the value is 0

We know that $\binom{r}{k} = \dfrac{r^{\underline{k}}}{k!}$

$$\Rightarrow \frac{(-1)^k (k-r-1)^{\underline{k}}}{k!}$$

$= (-1)^k \binom{k-r-1}{k}$ integer k; (we get this by taking r as k - r -1 in binomial coefficient formula) (Text Book Definition 5.14)

Now to get value $\binom{-1}{k}$ we take r = -1

$= (-1)^k \binom{k-(-1)-1}{k} = (-1)^k \binom{k}{k} = (-1)^k$ integer k ; (Since $\binom{k}{k} = 1$)

So the solution will be 1 or -1 depending on the value of k

$\binom{-1}{k} = (-1)^k$, where k is an integer greater than 0

When k<0 the value is 0

Problem 6: Fix up the text's derivation in Problem 6, Section 5.2, by correctly applying symmetry.

Problem 6 is to find a closed form for the summation:

$$\sum_{k>=0} \binom{n+k}{2k} \binom{2k}{k} \frac{(-1)^k}{k+1} \quad , \text{ integer } n>=0$$

Observation: In this sum the index of summation "k" appears six times. This can be tricky to solve the summation. But the 2k's in the summation appear in a way that we can make use the identity to simplify the product of two binomial coefficients.

$$\binom{r}{m} \binom{m}{k} = \binom{r}{k} \binom{r-k}{m-k}$$

where m, k are integers.

Applying the identity to our summation we get:

$$\sum_{k>=0} \binom{n+k}{2k} \binom{2k}{k} \frac{(-1)^k}{k+1} = \sum_{k>=0} \binom{n+k}{k} \binom{n}{k} \frac{(-1)^k}{k+1} \rightarrow (1)$$

(Here r=n+k, m=2k and k=k)

This step reduces the occurrence of “k” by 1. So now we have 5 more to go.

The k+1 factor in this above summation can be absorbed into $\binom{n}{k}$ by using the following identity:

$$\boxed{\binom{r}{k} = \frac{r}{k} \binom{r-1}{k-1}}$$

This gives us: $\binom{n+1}{k+1} = \frac{n+1}{k+1} \binom{n}{k}$

Rearranging terms we get: $\binom{n}{k} = \frac{k+1}{n+1} \binom{n+1}{k+1}$

Let us use this value of $\binom{n}{k}$ in our summation equation (1).

Hence we get :

$$\sum_{k>=0} \binom{n+k}{k} \binom{n}{k} \frac{(-1)^k}{k+1} = \sum_{k} \binom{n+k}{k} \binom{n+1}{k+1} \frac{(-1)^k}{k+1} \frac{k+1}{n+1}$$

$$= \sum_{k} \binom{n+k}{k} \binom{n+1}{k+1} \frac{(-1)^k}{n+1}$$

$$= \frac{1}{n+1} \sum_{k} \binom{n+k}{k} \binom{n+1}{k+1} (-1)^k$$

→ (2)

(Here n>=0)

Now we have got rid of 2 occurrences of "k". To eliminate another k we can either use symmetry on $\binom{n+k}{k}$ or negate the upper index n+k which eliminates k and also the factor $(-1)^k$.

The rule of symmetry is as follows:

$$\boxed{\binom{n}{k} = \binom{n}{n-k}}$$

The textbook uses symmetry identity on equation (2) as follows:

$$\frac{1}{n+1} \sum_k \binom{n+k}{k}\binom{n+1}{k+1}(-1)^k = \frac{1}{n+1} \sum_k \binom{n+k}{n}\binom{n+1}{k+1}(-1)^k \quad \rightarrow (3)$$

This eliminates another occurrence of "k" in the summation.

Now we can apply the identity for sums of products of binomial coefficients to equation (3). The identity we can make use of here is:

$$\boxed{\sum_k \binom{l}{m+k}\binom{s+k}{n}(-1)^k = (-1)^{l+m}\binom{s-m}{n-l}}$$

(l,m,n are integers and l>=0)

Replacing (l,m,n,s) in this identity with (n+1,1,n,n) we get:

$$\frac{1}{n+1} \sum_k \binom{n+k}{n}\binom{n+1}{k+1}(-1)^k = \frac{1}{n+1}(-1)^n\binom{n-1}{-1} = \mathbf{0} \quad \rightarrow (4)$$

$\left(\text{Using the fact } \binom{n}{k} = 0 \text{ when } k<0 \right)$

Let us check the solution obtained through symmetry taking n=2.

Equation (4) now yields:

$$\binom{2}{0}\binom{2}{0}\frac{1}{1} - \binom{3}{2}\binom{2}{1}\frac{1}{2} + \binom{4}{4}\binom{4}{2}\frac{1}{3}$$

$$= \quad 1 \quad - \frac{6}{2} + \frac{6}{3} \quad = \quad \mathbf{0}$$

→ This example results in the same solution as in Equation (4)

Now let us explore the second option of negating the upper index of $\binom{n+k}{k}$ in Equation (2)

The identity for upper negation is given by:

$$\boxed{\binom{r}{k} = (-1)^k \binom{k-r-1}{k}}$$

Using this identity in Equation (4) we get:

$$\frac{1}{n+1} \sum_{k} \binom{n+k}{k}\binom{n+1}{k+1}(-1)^k = \frac{1}{n+1} \sum_{k} \binom{-n-1}{k}\binom{n+1}{k+1}$$

→ (5)

To the Equation (5) we can apply the following identity:

$$\sum_k \binom{l}{m+k}\binom{s}{n+k} = \binom{l+s}{l-m+n}$$

Apply this to Equation (5) with (l,m,n,s) ← (n+1,1,0,-n-1). This gives :

$$\frac{1}{n+1} \sum_k \binom{-n-1}{k}\binom{n+1}{k+1} = \frac{1}{n+1}\binom{0}{n}$$

This results in a 0 when n>0. But when n=0, we get $\frac{1}{n+1}\binom{0}{n} = \frac{1}{1}\binom{0}{0} = 1$

This is a **CONTRADICTION** to the result obtained by using symmetry on $\binom{n+k}{k}$

The result was 0 for all cases. But negating the upper index n+k, the result is 0 when n>0 and 1 when n=0.

The mistake happened when we applied symmetry when the upper index could be negative.

$\binom{n+k}{k}$ cannot be replaced by $\binom{n+k}{n}$ when “k” ranges over all integers since it converts 0 into a non-zero value when k<-n.

Also when k<-n, the factor $\binom{n+1}{k+1}$ becomes 0. The only exception is when n=0 and k = -1. Hence the result becomes different and this did not turn up when we used the example with n=2.

So to rectify mistake done while using symmetry on Equation (2), we need to consider the case when n=0 and k=-1. Applying this to our symmetry we get:

$$\frac{1}{n+1}\sum_{k}\binom{n+k}{k}\binom{n+1}{k+1}(-1)^k = \frac{1}{n+1}\sum_{k>=0}\binom{n+k}{n}\binom{n+1}{k+1}(-1)^k$$

$$= \frac{1}{n+1}\sum_{k}\binom{n+k}{n}\binom{n+1}{k+1}(-1)^k$$

$$- \frac{1}{n+1}\binom{n-1}{n}\binom{n+1}{0}(-1)^{-1}$$

From Equation (4) we know that the first term in the above expression is 0.

So : $\frac{1}{n+1}\sum_{k}\binom{n+k}{k}\binom{n+1}{k+1}(-1)^k = -\frac{1}{n+1}\binom{n-1}{n}\binom{n+1}{0}(-1)^{-1}$

Continued….

Continued….

$$\frac{1}{n+1}\sum_{k}\binom{n+k}{k}\binom{n+1}{k+1}(-1)^k = \frac{1}{n+1}\binom{n-1}{n}\binom{n+1}{0}$$

$$= \frac{1}{n+1}\binom{n}{-1}\binom{n+1}{0} \rightarrow \text{Closed Formula}$$

Hence we get the closed formula as :

$$\boxed{\sum_{k>=0}\binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{k+1} = \frac{1}{n+1}\binom{n}{-1}\binom{n+1}{0}}$$

Verification :

This sum is 0 for all values of n other than for n=0. And for n=0 we get the sum as 1.

Hence the closed formula for the summation is right.