Chapter 5 Problem 2

0011

Question 2

For which value(s) of k is $\binom{n}{k}$ a maximum, when n is a given positive integer?

Prove your answer.

Observation

	$k=0$	1	2	3	4	5	6
$n=1$	1	1					
$n=2$	1	2	1				
$n=3$	1	3	3	1			
$n=4$	1	4	6	4	1		
$n=5$	1	5	10	10	5	1	
$n=6$	1	6	15	20	15	6	1

$$\text{When } n = \text{odd, } k = \left\lceil \frac{n}{2} \right\rceil \text{ \& } k = \left\lfloor \frac{n}{2} \right\rfloor$$

$$\text{When } n = \text{even, } k = \frac{n}{2}$$

Proof :

First, let's assume $\binom{n}{k^*}$ is the maximum, where n $\in □^{+}$

Since $\binom{n}{k^*}$ is the maximum,

CASE I: $\binom{n}{k^*+1} \leq \binom{n}{k^*}$

CASE II: $\binom{n}{k^*-1} \leq \binom{n}{k^*}$

CASE I : $\binom{\mathbf{n}}{\mathbf{k^*+1}} \le \binom{\mathbf{n}}{\mathbf{k^*}}$

$$\binom{n}{k^*+1} = \frac{n!}{(n-k^*-1)!(k^*+1)!} = \frac{n!}{(n-k^*-1)!(k^*+1)k^*!} \quad \text{...... (A)}$$

$$\binom{n}{k^*} = \frac{n!}{k^*!(n-k^*)!}$$

$$= \frac{n!}{(n-k^*-1)!(k^*+1)k^*!} \times \frac{k^*+1}{(n-k^*)} \quad \text{...... (B)}$$

Substitute (A) and (B) into $\binom{n}{k^*+1} \le \binom{n}{k^*}$, we get:

$$\frac{n!}{(n-k^*-1)!(k^*+1)k^*!} \le \frac{n!}{(n-k^*-1)!(k^*+1)k^*!} \times \frac{k^*+1}{(n-k^*)}$$

In order for the above statement to be true, $\frac{k^*+1}{(n-k^*)} \ge 1$

$$\frac{k^*+1}{(n-k^*)} \ge 1$$

$$k^* - 1 \ge n - k^*$$

$$2k^* \ge n - 1$$

$$k^* \ge \frac{n-1}{2} \qquad \text{...... (1)}$$

CASE II: $\binom{\mathbf{n}}{\mathbf{k^*-1}} \le \binom{\mathbf{n}}{\mathbf{k^*}}$

$$\binom{n}{k^*-1} = \frac{n!}{(k^*-1)!(n-k^*+1)!} = \frac{n!}{(k^*-1)!(n-k^*)!(n-k^*+1)} \quad \text{...... (C)}$$

$$\binom{n}{k^*} = \frac{n!}{k^*!(n-k^*)!}$$

$$= \frac{n!}{(k^*-1)!(n-k^*)!(n-k^*+1)} \times \frac{(n-k^*+1)}{k^*} \quad \text{...... (D)}$$

Substitute (C) and (D) into $\binom{n}{k^*-1} \le \binom{n}{k^*}$, we get:

$$\frac{n!}{(k^*-1)!(n-k^*)!(n-k^*+1)} \le \frac{n!}{(k^*-1)!(n-k^*)!(n-k^*+1)} \times \frac{(n-k^*+1)}{k^*}$$

In order for the above statement to be true, $\frac{(n-k^*+1)}{k^*} \ge 1$

$$\frac{(n-k^*+1)}{k^*} \ge 1$$

$$n-k^*+1 \ge k^*$$

$$n+1 \ge 2k^*$$

$$\frac{n+1}{2} \ge k^* \qquad \text{...... (2)}$$

From (1) & (2) we derived,

$$\frac{n-1}{2} \le k^* \le \frac{n+1}{2}$$

To evaluate this further:

When n is even, $n = 2a,\ a \in \mathbb{Z}^+$

$$\frac{n-1}{2} \le k^* \le \frac{n+1}{2}$$

$$\frac{2a-1}{2} \le k^* \le \frac{2a+1}{2}$$

since $k^* \in \mathbb{Z}$ between $\dfrac{2a-1}{2}$ & $\dfrac{2a+1}{2} \notin \mathbb{Z}$

$$k^* = \frac{2a}{2} = \frac{n}{2}$$

When n is odd: $$\frac{n-1}{2} \le k^* \le \frac{n+1}{2}$$

(3.5b) from the textbook states:

$$\lfloor x^* \rfloor = n^* \ \ iff\ \ x^* - 1 < n^* \le x^*\ ,\ n^* \in \mathbb{Z}$$

Using this property,

$$\frac{n}{2} - 1 < \frac{n-1}{2} \le \frac{n}{2}$$

where $x^* = \frac{n}{2},\ n^* = \frac{n-1}{2}$

$$\frac{n-1}{2} = \left\lfloor \frac{n}{2} \right\rfloor$$

(3.5d) from the textbook states:

$$\lceil x^* \rceil = n^* \ \ iff\ \ x^* < n^* \le x^* + 1\ ,\ n^* \in \mathbb{Z}$$

Using this property,

$$\frac{n}{2} < \frac{n+1}{2} \le \frac{n}{2} + 1$$

where $x^* = \frac{n}{2},\ n^* = \frac{n+1}{2}$

$$\frac{n+1}{2} = \left\lceil \frac{n}{2} \right\rceil$$

In conclusion :

$$\frac{n-1}{2} \le k^* \le \frac{n+1}{2}$$

$$\left\lfloor \frac{n}{2} \right\rfloor \le k^* \le \left\lceil \frac{n}{2} \right\rceil$$

SOLUTION!!!

In fact, this include the k^* when n is even.