

# Chapter 5 Problem 2

0011



## Question 2

For which value(s) of  $k$  is  $\binom{n}{k}$  a maximum,  
when  $n$  is a given positive integer?

Prove your answer.

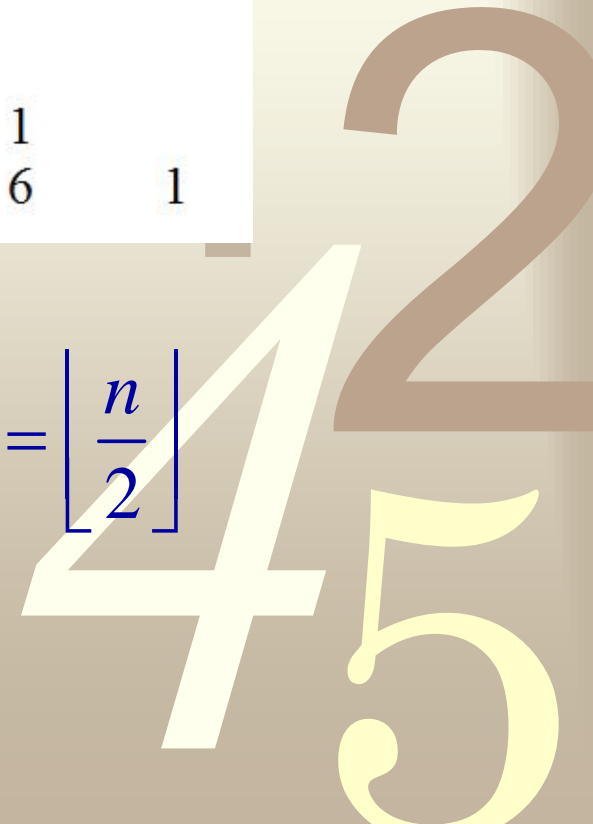


# Observation

	$k=0$	1	2	3	4	5	6
$n=1$	1	1					
$n=2$	1	2	1				
$n=3$	1	3	3	1			
$n=4$	1	4	6	4	1		
$n=5$	1	5	10	10	5	1	
$n=6$	1	6	15	20	15	6	1

When  $n = \text{odd}$ ,  $k = \left\lceil \frac{n}{2} \right\rceil$  &  $k = \left\lfloor \frac{n}{2} \right\rfloor$

When  $n = \text{even}$ ,  $k = \frac{n}{2}$



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## Proof :

First, let's assume  $\binom{n}{k^*}$  is the maximum, where  $n \in \mathbb{N}^+$

Since  $\binom{n}{k^*}$  is the maximum,

$$\text{CASE I: } \binom{n}{k^* + 1} \leq \binom{n}{k^*}$$

$$\text{CASE II: } \binom{n}{k^* - 1} \leq \binom{n}{k^*}$$

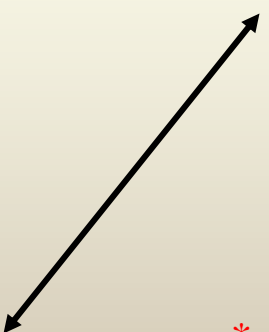


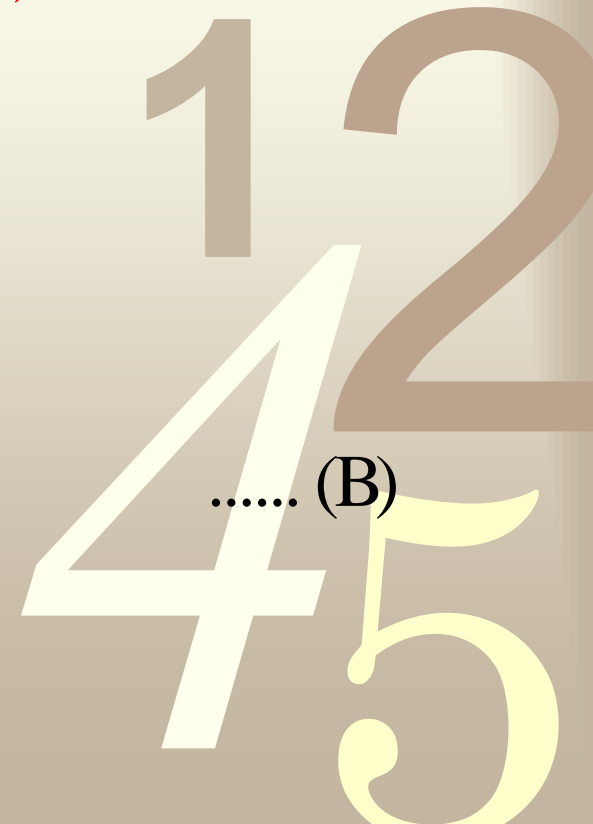
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**CASE I:**  $\binom{n}{k^*+1} \leq \binom{n}{k^*}$

$$\binom{n}{k^*+1} = \frac{n!}{(n-k^*-1)!(k^*+1)!} = \frac{n!}{(n-k^*-1)!(k^*+1)k^*!} \quad \dots (A)$$

$$\binom{n}{k^*} = \frac{n!}{k^*!(n-k^*)!}$$

$$= \frac{n!}{(n-k^*-1)!(k^*+1)k^*!} \times \frac{k^*+1}{(n-k^*)} \quad \dots (B)$$




Substitute (A) and (B) into  $\binom{n}{k^*+1} \leq \binom{n}{k^*}$ , we get:

$$\frac{n!}{(n-k^*-1)!(k^*+1)k^*!} \leq \frac{n!}{(n-k^*-1)!(k^*+1)k^*!} \times \frac{k^*+1}{(n-k^*)}$$

In order for the above statement to be true,  $\frac{k^*+1}{(n-k^*)} \geq 1$

$$\frac{k^*+1}{(n-k^*)} \geq 1$$

$$k^*-1 \geq n-k^*$$

$$2k^* \geq n-1$$

$$k^* \geq \frac{n-1}{2}$$

..... (1)

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**CASE II:**  $\binom{n}{k^* - 1} \leq \binom{n}{k^*}$

$$\binom{n}{k^* - 1} = \frac{n!}{(k^* - 1)!(n - k^* + 1)!} = \frac{n!}{(k^* - 1)!(n - k^*)!(n - k^* + 1)} \quad \dots (C)$$

$$\binom{n}{k^*} = \frac{n!}{k^*!(n - k^*)!}$$

$$= \frac{n!}{(k^* - 1)!(n - k^*)!(n - k^* + 1)} \times \frac{(n - k^* + 1)}{k^*} \quad \dots (D)$$



Substitute (C) and (D) into  $\binom{n}{k^* - 1} \leq \binom{n}{k^*}$ , we get:

$$\frac{n!}{(k^* - 1)!(n - k^*)!(n - k^* + 1)} \leq \frac{n!}{(k^* - 1)!(n - k^*)!(n - k^* + 1)} \times \frac{(n - k^* + 1)}{k^*}$$

In order for the above statement to be true,  $\frac{(n - k^* + 1)}{k^*} \geq 1$

$$\frac{(n - k^* + 1)}{k^*} \geq 1$$

$$n - k^* + 1 \geq k^*$$

$$n + 1 \geq 2k^*$$

$$\frac{n + 1}{2} \geq k^*$$

..... (2)



From (1) & (2) we derived,

$$\frac{n-1}{2} \leq k^* \leq \frac{n+1}{2}$$

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To evaluate this further:

When  $n$  is even,  $n = 2a$ ,  $a \in \mathbb{N}^+$

$$\frac{n-1}{2} \leq k^* \leq \frac{n+1}{2}$$

$$\frac{2a-1}{2} \leq k^* \leq \frac{2a+1}{2}$$

since  $k^* \in \mathbb{N}$  between  $\frac{2a-1}{2}$  &  $\frac{2a+1}{2} \notin \mathbb{N}$

$$k^* = \frac{2a}{2} = \frac{n}{2}$$

When  $n$  is odd:

$$\frac{n-1}{2} \leq k^* \leq \frac{n+1}{2}$$

(3.5b) from the textbook states:

$$\lfloor x^* \rfloor = n^* \text{ iff } x^* - 1 < n^* \leq x^*, n^* \in \mathbb{Z}$$

Using this property,

$$\frac{n}{2} - 1 < \frac{n-1}{2} \leq \frac{n}{2}$$

$$\text{where } x^* = \frac{n}{2}, n^* = \frac{n-1}{2}$$

$$\frac{n-1}{2} = \left\lfloor \frac{n}{2} \right\rfloor$$

(3.5d) from the textbook states:

$$\lceil x^* \rceil = n^* \text{ iff } x^* < n^* \leq x^* + 1, n^* \in \mathbb{Z}$$

Using this property,

$$\frac{n}{2} < \frac{n+1}{2} \leq \frac{n}{2} + 1$$

$$\text{where } x^* = \frac{n}{2}, n^* = \frac{n+1}{2}$$

$$\frac{n+1}{2} = \left\lceil \frac{n}{2} \right\rceil$$

# In conclusion :

$$\frac{n-1}{2} \leq k^* \leq \frac{n+1}{2}$$

$$\left\lfloor \frac{n}{2} \right\rfloor \leq k^* \leq \left\lceil \frac{n}{2} \right\rceil$$

**SOLUTION!!!**

In fact, this include the  $k^*$  when  $n$  is even.

