CSE547 Discrete Mathematics
Chapter 3, problem 23

Problem Statement

Show that the nth element of the sequence:
1,2,2,3,3,3,4,4,4,4,5,5,5,5,5,…
is $\left\lfloor \sqrt{2n} + \frac{1}{2} \right\rfloor$

Let the n^{th} element be m. We need to find m

Let P(x) represent the position of the last occurrence of x in the sequence.

We show that $P(x) = \frac{x(x+1)}{2}$ by induction:

Basis: It is true for P(1) since 1(1+1)/2 = 1

Since x is (x) places ahead of last occurrence of x-1, we have:

$$P(x) = P(x-1) + x$$

$$= \frac{(x-1)x}{2} + x$$ by inductive assumption

$$= \frac{(x-1)x + 2x}{2}$$

$$= \frac{(x - 1 + 2)x}{2}$$

$$= \frac{x(x + 1)}{2}$$

Therefore $P(x) = \frac{x(x+1)}{2}$ for all x Є Z^+

Hence proved.

$$P(m-1) = \frac{(m-1)m}{2}$$

$$P(m) = \frac{m(m+1)}{2}$$

Since the n^{th} element should lie between P(m-1) and P(m), we have:

$$\frac{(m-1)m}{2} < n <= \frac{m(m+1)}{2}$$

$(m-1)m < 2n < m(m+1) < m(m+1) + 1/4$

obvious

$$m^2 + m + \frac{1}{4}$$

is strictly greater than 2n

$$m^2 - m + \frac{1}{4} \quad <= \quad 2n \quad < \quad m^2 + m + \frac{1}{4}$$

We add ¼ -does not give next integer, but we can put <= 2n

$$\left(m - \frac{1}{2}\right)^2 <= 2n < \left(m + \frac{1}{2}\right)^2$$

$$m - \frac{1}{2} \quad <= \sqrt{2n} \quad < \quad m + \frac{1}{2}$$

Add ½ to all sides

$$m = \left\lfloor \sqrt{2n} + \frac{1}{2} \right\rfloor$$

Follows from of (3.5)(a) which is:

$$\lfloor x \rfloor = n \Leftrightarrow n \leq x < n + 1$$

Thank you