CSE547

- Chapter 3, problems 19, 20

Problem 20

- Problem Statement

  Find the sum of all multiples of x in the closed interval [α.. β], when $x > 0$.

Problem: Find the sum of all multiples of x in the closed interval [α.. β], when x > 0.

Let

S = Sum of all the multiples of x in the interval [α.. β]

b $\sum_{k \varepsilon Z,\ \alpha \le kx \le \beta} kx$

=

$$\sum kx\,[\,k \text{ is an integer}]\,[\alpha \le kx \le \beta\,]$$

=

(Rewriting in Iversonian Form)

$$x \sum k\,[\,k \text{ is an integer}]\,[\alpha \le kx \le \beta\,]$$

=

(Since x is a constant.)

Problem: Find the sum of all multiples of x in the closed interval [α.. β], when x > 0.

$$= \quad x \sum k \, [\, k \text{ is an integer}] \, [\alpha/x \;\le\; k \;\le\; \beta/x \,]$$

(Since x > 0)

$$= \quad x \sum k \, [\, k \text{ is an integer}] \, [\, \lceil \alpha/x \rceil \;\le\; k \;\le\; \lfloor \beta/x \rfloor \,]$$

(Since,

$$x \le n \quad \leftrightarrow \quad \lceil x \rceil \le n$$

$$n \le x \quad \leftrightarrow \quad n \;\le\; \lfloor x \rfloor$$

where n is an integer, x is a real)

Problem: Find the sum of all multiples of x in the closed interval [α.. β], when x > 0.

$$= \; x\sum k \, [\, k \text{ is an integer}] \, [\, \left\lceil \frac{\alpha}{x} \right\rceil \le \; k \; < \; \left\lfloor \frac{\beta}{x} \right\rfloor + 1 \,]$$

$$= \quad x\sum_{\left\lceil \frac{\alpha}{x} \right\rceil}^{\left\lfloor \frac{\beta}{x} \right\rfloor + 1} k^{\underline{1}} \, \delta k$$

We know that,

$$\sum k^{\underline{1}} \delta k \quad = \quad \frac{k^{\underline{2}}}{2}$$

Problem: Find the sum of all multiples of x in the closed interval [α.. β], when x > 0.

$$S = x \sum_{\left\lceil \frac{\alpha}{x} \right\rceil}^{\left\lfloor \frac{\beta}{x} \right\rfloor + 1} k^{\underline{1}} \, \delta k$$

$$= \frac{x k^{\underline{2}}}{2} \Big|_{\left\lceil \frac{\alpha}{x} \right\rceil}^{\left\lfloor \frac{\beta}{x} \right\rfloor + 1}$$

Thus, the sum of all multiples of x in the closed interval [α.. β], when x > 0.

$$= \frac{x k^{\underline{2}}}{2} \Big|_{\left\lceil \frac{\alpha}{x} \right\rceil}^{\left\lfloor \frac{\beta}{x} \right\rfloor + 1}$$

Problem 19

- Problem Statement

Find a necessary and sufficient condition on the real number b > 1 such that $\lfloor \log_b \lfloor x \rfloor \rfloor$ **=** $\lfloor \log_b x \rfloor$ **for all real x ≥ 1.**

A bit more formally, the question is:

Define a predicate $P: \{y \in R: y > 1\} \rightarrow \{\text{true}, \text{false}\}$, such that:

I. $\forall_{b \in R, b>1} (P(b) \Rightarrow \forall_{x \in R, x \geq 1} (\lfloor \log_b \lfloor x \rfloor \rfloor = \lfloor \log_b x \rfloor))$ (sufficiency)

II. $\forall_{b \in R, b>1} (\forall_{x \in R, x \geq 1} (\lfloor \log_b \lfloor x \rfloor \rfloor = \lfloor \log_b x \rfloor) \Rightarrow P(b))$ (necessity)

Problem: Find a necessary and sufficient condition on the real number b > 1 such that $\lfloor \log_b \lfloor x \rfloor \rfloor = \lfloor \log_b x \rfloor$ for all real $x \geq 1$.

- We will prove that the predicate P defined on $\{y \in R: y > 1\} \rightarrow$ { true, false } with value $P(y) = y \in Z$ satisfies (i) and (ii).

- <u>Proof of Sufficiency Condition:</u>

- We have that b ϵ R, b > 1, and P(b). P(b) being true gives us that b ϵ Z.

- Now define a function f: $\{x \in R: x \geq 1\} \rightarrow R$

  f(x) = $\log_b x$

- Since b > 1, f(x) is continuous and monotonically increasing on $[1, \infty)$, these are known properties of the log function.

Problem: Find a necessary and sufficient condition on the real number b > 1 such that

$\lfloor \log_b \lfloor x \rfloor \rfloor = \lfloor \log_b x \rfloor$ for all real x ≥ 1.

- We can prove that **f(x) ϵ Z ⇒ x ϵ Z**
- Note:

  f(x) ε Z, by assumption

  $\log_b x$ ε Z, by substitution

- $\log_b x \geq 0$, since $\log_b 1 = 0$, 1 is the smallest element of the domain of f, and f is monotonically increasing.

- $b^{\log_b x}$ ε Z, since b is a positive integer and $\log_b x$ is a nonnegative integer.

  If $\log_b x$ is 0 then $b^{\log_b x} = 1$. Else $\log_b x$ is a positive integer, $b^{\log_b x}$ ϵ Z^+

since positive integers are closed under exponentiation.

- Simplifying, we have $b^{\log_b x}$ = x. Hence x ε Z

Problems: Find a necessary and sufficient condition on the real number $b > 1$ such that $\lfloor \log_b \lfloor x \rfloor \rfloor = \lfloor \log_b x \rfloor$ for all real $x \geq 1$.

- Therefore $f(x)$ is continuous, $f(x)$ is monotonically increasing, and $f(x) \in Z \Rightarrow x \in Z$. Therefore we can invoke Theorem 3.10 proven in the textbook :

> If f is a continuous, monotonically increasing function with the property that $f(x) \in Z \Rightarrow x \in Z$, then $\lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor$ whenever $f(x)$ and $f(\lfloor x \rfloor)$ are defined.

- Our $f(x)$ is defined on $[1, \infty)$, so when $x \geq 1$, $f(x)$ is defined. Additionally, when $x \geq 1$, $\lfloor x \rfloor \geq 1$ (by property 3.7 d), so $f(\lfloor x \rfloor)$ is defined.

- Therefore we get that $\forall_{x \in R, x \geq 1} (\lfloor \log_b \lfloor x \rfloor \rfloor = \lfloor \log_b x \rfloor)$
- this completes part I.

Problem: Find a necessary and sufficient condition on the real number b > 1 such that
$\lfloor \log_b \lfloor x \rfloor \rfloor = \lfloor \log_b x \rfloor$ for all real x ≥ 1.

- <u>Proof of Necessary Condition</u>

- We need to prove that $\forall_{b \in R, b>1} (\forall_{x \in R, x \geq 1}(\lfloor \log_b \lfloor x \rfloor \rfloor = \lfloor \log_b x \rfloor) \Rightarrow P(b))$

- We'll prove it by contradiction. So assume:

$$\neg (\forall_{b \in R, b>1} (\forall_{x \in R, x \geq 1}(\lfloor \log_b \lfloor x \rfloor \rfloor = \lfloor \log_b x \rfloor) \Rightarrow P(b)))$$

- This means: $\exists_{b \in R, b>1} (\forall_{x \in R, x \geq 1}(\lfloor \log_b \lfloor x \rfloor \rfloor = \lfloor \log_b x \rfloor) \not\Rightarrow P(b))$

- Which means: $\exists_{b \in R, b>1} (\forall_{x \in R, x \geq 1}(\lfloor \log_b \lfloor x \rfloor \rfloor = \lfloor \log_b x \rfloor) \wedge \neg P(b))$

- 1. b $\in R$
  2. b > 1
  3. $\forall_{x \in R, x \geq 1}(\lfloor \log_b \lfloor x \rfloor \rfloor = \lfloor \log_b x \rfloor)$
  4. $\neg P(b)$

Problem: Find a necessary and sufficient condition on the real number b > 1 such that

$\lfloor \log_b \lfloor x \rfloor \rfloor = \lfloor \log_b x \rfloor$ for all real x ≥ 1.

- Note that (1), (2), (3) imply that $\lfloor \log_b \lfloor b \rfloor \rfloor = \lfloor \log_b b \rfloor$

- We know that $\log_b b = 1$, so this gives us

  $\lfloor \log_b \lfloor b \rfloor \rfloor = \lfloor 1 \rfloor = 1$

- Now, using (4) from earlier, $\neg P(b) \Rightarrow b \notin Z$

- Now, $\lfloor b \rfloor \le b$ of course, and since $\lfloor b \rfloor \in Z$, but $b \notin Z$, $\lfloor b \rfloor < b$
- so $\lfloor b \rfloor \ne b$

- Then because the log function is monotonically increasing, we have

  $\log_b \lfloor b \rfloor < \log_b b$

- We have $\log_b b = 1$, so we get $\log_b \lfloor b \rfloor < 1$

Problem: Find a necessary and sufficient condition on the real number b > 1 such that $\lfloor \log_b \lfloor x \rfloor \rfloor = \lfloor \log_b x \rfloor$ for all real x ≥ 1.

- By property 3.7a then, $\lfloor \log_b \lfloor b \rfloor \rfloor < 1$, which contradicts the previously derived
- This completes the proof of part II.
- Thus, we have proved that:

  The necessary and sufficient condition on the real number b > 1 such that $\lfloor \log_b \lfloor x \rfloor \rfloor = \lfloor \log_b x \rfloor$, for all real x ≥ 1.

  is that $b \in Z$

That is, **b must be an integer.**