Cse547 Discrete Mathematics
Chapter 3, Problem 16

Problem 16:

Prove that n mod 2 = $(1 - (-1)^n)/ 2$.

Find and prove a similar expression for n mod 3 in the form

$a + b\,\omega^n + c\,\omega^{2n}$, where ω is the complex number $(-1 + i\sqrt{3}) / 2$.

Hint: $\omega^3 = 1$ and $(1 + \omega + \omega^2) = 0$

Part 1: Prove the given formula for n mod 2

<u>When n is odd:</u>

We know that: n mod 2 = 1

$(-1)^n = -1$ since n is odd

$$\frac{(1-(-1)^n)}{2} = \frac{(1-(-1))}{2} = \frac{2}{2} = 1$$

Therefore,

$$n \bmod 2 = \frac{(1-(-1)^n)}{2}$$ when 'n' is odd.

When n is even:

We know that: $n \bmod 2 = 0$

$(-1)^n = 1$ since n is even

$$\frac{(1-(-1)^n)}{2} = \frac{(1-(1))}{2} = \frac{0}{2} = 0$$

Therefore,

$$n \bmod 2 = \frac{(1-(-1)^n)}{2}$$ when 'n' is even.

With those 2 cases we can conclude that –

$$n \bmod 2 = \frac{(1 - (-1)^n)}{2} \quad , \; n \in \mathbb{Z}$$

Part 2: Finding expression for n mod 3

- We have to find expression in the following form

$$n \bmod 3 = a + b\,\omega^{n} + c\,\omega^{2n}$$

$$\text{where} \quad \omega = \frac{(-1 + i\sqrt{3})}{2}$$

- Essentially, we need to find values of a, b, c.
- We need at least 3 equations for 3 variables.

Observe that directly from the definition we have that

If (n mod 3) $\in$ M, M is some set

Then,

$$M = \{0, 1, 2\}$$

When n = 0, (n mod 3) = 0

$a + b\,\omega^n + c\,\omega^{2n} \;=\; a + b + c \;=\; 0$ .... [1]

When n = 1, (n mod 3) = 1

$a + b\,\omega^n + c\,\omega^{2n} \;=\; a + b\,\omega + c\,\omega^2 \;=\; 1$

$= a + b\,\omega + c\,(-\omega -1) \;=\; 1$ since $(\omega 2 + \omega + 1 = 0)$

$= a + (b - c)\,\omega - c - 1 \;=\; 0$ .... [2]

When n = 2, (n mod 3) = 2

$a + b\,\omega^n + c\,\omega^{2n} \;=\; a + b\,\omega^2 + c\,\omega^4 \;=\; 2$

$= a + b\,(-\omega-1) + c\,\omega^3\,\omega \;=\; 2$ since $(\omega 2 + \omega + 1 = 0)$

$= a - b\,\omega - b + c\,\omega \;=\; 2$ since $\omega^3 = 1$

$= a - b - (b - c)\,\omega - 2 \;=\; 0$ .... [3]

We have (1), (2) and (3) – 3 equations and 3 unknowns – a, b, c. Solve it. ☺

Adding equations [1] , [2] & [3]

$$[1] + [2] + [3] \Rightarrow$$

$$[a + b + c] + [a + (b - c)\,\omega - c - 1] + [a - b - (b - c)\,\omega - 2] = 0$$

$$3a - 3 = 0$$

$$a = 1$$

Substituting a = 1 in [1] and [3] we get,

[1]: $a + b + c = 0$

$1 + b + c = 0$ .... [4]

[3]: $a - b - (b - c)\,\omega - 2 = 0$

$1 - b - (b - c)\,\omega - 2 = 0$

$-1 - b - (b - c)\,\omega = 0$ .... [5]

[4] x ω => $\omega + b\,\omega + c\,\omega = 0$ .... [6]

[5] – [6] => $\{-1 - b - (b-c)\,\omega\} - \{\omega + b\,\omega + c\,\omega\} = 0$

$-1 - b - b\,\omega + c\,\omega - \omega - b\,\omega - c\,\omega = 0$ [Got rid of c]

$2b\,\omega + b + \omega + 1 = 0$

$$b = \frac{-\omega - 1}{2\,\omega + 1}$$

Simplifying b further :-

$$b = \frac{-\omega - 1}{2\omega + 1} = \frac{-\omega - 1}{2\omega + 1} \times \frac{(\omega - 1)}{(\omega - 1)}$$

$$b = \frac{-(\omega + 1)\ (\omega - 1)}{2\omega^2 - \omega - 1}$$

$$b = \frac{-(\omega + 1)(\omega - 1)}{2(-\omega - 1) - \omega - 1} \qquad \text{since } (\omega^2 + \omega + 1 = 0)$$

$$b = \frac{-(\omega + 1)(\omega - 1)}{-3 - 3\omega}$$

$$\boxed{b = \frac{(\omega + 1)(\omega - 1)}{3(\omega + 1)} = \frac{(\omega - 1)}{3}}$$

Substituting value of b in equation [4] to get value of c:

[4]: $1 + b + c = 0$

$c = -b - 1$

$$c = \frac{-(\omega - 1)}{3} - 1$$

$$c = \frac{-\omega + 1 - 3}{3}$$

$$\boxed{c = \frac{-(\omega + 2)}{3}}$$

So the expression is like this:-

$$n \bmod 3 = 1 + \frac{1}{3}\{(\omega - 1)\omega^{n} - (\omega + 2)\omega^{2n}\}$$