CSE547: Discrete Mathematics

Chapter 3: Problems 10, 12

Problem 3.10

Show that the expression

$$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor$$

is always either $\lceil x \rceil$ or $\lfloor x \rfloor$.

In what circumstances does each case arise?

Definitions

Floor, $\lfloor x \rfloor$ = the greatest integer less or equal x

Ceiling, $\lceil x \rceil$ = the least integer less or equal x

Example:

e~2.718, therefore $\lfloor e \rfloor$ = 2 and $\lceil e \rceil$ = 3

Different Cases

Let's re-write the expression as follows:

$$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor = \left\lceil x + \frac{1}{2} \right\rceil - \left(\left\lceil \frac{2x+1}{4} \right\rceil - \left\lfloor \frac{2x+1}{4} \right\rfloor\right)$$

Let $\frac{2x+1}{4} = n$

If <u>x is an integer</u>, then $\frac{2x+1}{4} = (\frac{x}{2} + \frac{1}{4})$ **= <u>n is not an integer</u>**

Example: $x = 1;\ \frac{2x+1}{4} = \frac{3}{4}$ $\qquad$ $x = 2;\ \frac{2x+1}{4} = \frac{5}{4}$

If <u>x is not an integer</u>, then $\frac{2x+1}{4} = (\frac{x}{2} + \frac{1}{4})$ **= <u>n is an integer OR not an integer</u>**

Example: $x = 3.5;\ \frac{2x+1}{4} = 8/4 = 2$ $\qquad$ **OR** $\qquad$ $x = 2.8;\ \frac{2x+1}{4} = 6.6/4 = 1.65$

Therefore, we will consider 2 cases:

- **n is an integer**
- **n is not an integer**

Case 1: $\frac{2x+1}{4} = n$ - is an integer

Evaluate: $\lceil x + \frac{1}{2} \rceil - (\lceil \frac{2x+1}{4} \rceil - \lfloor \frac{2x+1}{4} \rfloor)$

Facts:

- $(\lceil \frac{2x+1}{4} \rceil - \lfloor \frac{2x+1}{4} \rfloor) = (\lceil n \rceil - \lfloor n \rfloor) = 0$ because $\lceil n \rceil = \lfloor n \rfloor$
- Since $\frac{2x+1}{4} = n$, then $x = 2n - \frac{1}{2}$, therefore $\lceil x \rceil = \lceil 2n - \frac{1}{2} \rceil = \lceil 2n \rceil$

Considering the above:

$$\lceil x + \frac{1}{2} \rceil - (\lceil \frac{2x+1}{4} \rceil - \lfloor \frac{2x+1}{4} \rfloor) = \lceil x + \frac{1}{2} \rceil - 0 = \lceil 2n - \frac{1}{2} + \frac{1}{2} \rceil = \lceil 2n \rceil = \lceil x \rceil$$

Therefore, for Case 1 we proved that:

$$\lceil \frac{2x+1}{2} \rceil - \lceil \frac{2x+1}{4} \rceil + \lfloor \frac{2x+1}{4} \rfloor = \lceil x \rceil$$

Case 2: $\frac{2x+1}{4} = n$ is not an integer

Evaluate: $\lceil x + \frac{1}{2} \rceil - (\lceil \frac{2x+1}{4} \rceil - \lfloor \frac{2x+1}{4} \rfloor)$

Formula (3.2) from the textbook:

$$\lceil y \rceil - \lfloor y \rfloor = [y \text{ is not an integer}] = 1$$

Therefore:

$$(\lceil \frac{2x+1}{4} \rceil - \lfloor \frac{2x+1}{4} \rfloor) = [\frac{2x+1}{4} \text{ is not an integer}] = 1$$

Formula (3.6) from the textbook:

$$\lceil y \rceil + n = \lceil y + n \rceil, \text{ where n is an integer}$$

Considering the above:

$$\lceil x + \frac{1}{2} \rceil - (\lceil \frac{2x+1}{4} \rceil - \lfloor \frac{2x+1}{4} \rfloor) = \lceil x + \frac{1}{2} \rceil - 1 = \lceil x + \frac{1}{2} - 1 \rceil = \lceil x - \frac{1}{2} \rceil$$

Continued on the next slide…

Case 2: $\frac{2x+1}{4} = n$ is not an integer

From the previous slide: $\lceil x + \frac{1}{2} \rceil - (\lceil \frac{2x+1}{4} \rceil - \lfloor \frac{2x+1}{4} \rfloor) = \lceil x - \frac{1}{2} \rceil$

Formula (3.8) from the textbook:

$$x = \lfloor x \rfloor + \{ x \} \text{ where } \lfloor x \rfloor \text{ is an integer and } 0 < \{ x \} < 1$$

Therefore:

$$\lceil x - \frac{1}{2} \rceil = \lceil \lfloor x \rfloor + \{ x \} - \frac{1}{2} \rceil$$

- **If** $\{ x \} > \frac{1}{2}$ **then** $\lceil \lfloor x \rfloor + \{ x \} - \frac{1}{2} \rceil = \lceil \lfloor x \rfloor + y \rceil = \lceil x \rceil$, **where** $0 < y < \frac{1}{2}$

- **If** $\{ x \} < \frac{1}{2}$ **then** $\lceil \lfloor x \rfloor + \{ x \} - \frac{1}{2} \rceil = \lceil \lfloor x \rfloor - y \rceil = \lfloor x \rfloor$, **where** $0 < y < \frac{1}{2}$

- **If** $\{ x \} = \frac{1}{2}$ **then** $\lceil \lfloor x \rfloor + \{ x \} - \frac{1}{2} \rceil = \lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$

Therefore, for Case 2 we proved that:

$$\lceil \frac{2x+1}{2} \rceil - \lceil \frac{2x+1}{4} \rceil + \lfloor \frac{2x+1}{4} \rfloor = \lceil x \rceil \text{ or } \lfloor x \rfloor$$

Problem 3.10: Conclusions

We showed that the expression

$$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor$$

is always either $\lceil x \rceil$ or $\lfloor x \rfloor$.

The formula gives an "unbiased" way to round.

Problem 3.12

Prove that

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor$$

for all integers n and all positive integers m.

This identity gives us another way to convert ceilings to floors and vice versa, instead of using the reflective law:

$$\lfloor -x \rfloor = - \lceil x \rceil$$
$$\lceil -x \rceil = - \lfloor x \rfloor.$$

Definitions

$$\frac{n}{m} = \left\lfloor \frac{n}{m} \right\rfloor + \left\{ \frac{n}{m} \right\}$$

Therefore $n = m\left\lfloor \frac{n}{m} \right\rfloor + (n \bmod m)$,

Where

$\left\lfloor \frac{n}{m} \right\rfloor$ **is a quotient**

(n mod m) is a remainder

Thus

$$(n \bmod m) = n - m\left\lfloor \frac{n}{m} \right\rfloor$$

Problem 3.12

Prove: $\lceil \frac{n}{m} \rceil = \lfloor \frac{n+m-1}{m} \rfloor$

Subtracting $\lfloor \frac{n}{m} \rfloor$ from both sides:

$$\lceil \frac{n}{m} \rceil - \lfloor \frac{n}{m} \rfloor = \lfloor \frac{n+m-1}{m} \rfloor - \lfloor \frac{n}{m} \rfloor$$

Lets use formula (3.6) from the textbook:

$\lfloor x + n \rfloor = \lfloor x \rfloor + n$, where n is integer

Considering the above:

$$\lceil \frac{n}{m} - \lfloor \frac{n}{m} \rfloor \rceil = \lfloor \frac{n+m-1}{m} - \lfloor \frac{n}{m} \rfloor \rfloor$$

since $\lfloor \frac{n}{m} \rfloor$ is an integer by definition

Problem 3.12

From the previous slide:

$$\left\lceil \frac{n}{m} - \left\lfloor \frac{n}{m} \right\rfloor \right\rceil = \left\lfloor \frac{n+m-1}{m} - \left\lfloor \frac{n}{m} \right\rfloor \right\rfloor$$

We can re-write the above as:

$$\left\lceil \frac{n - m\left\lfloor \frac{n}{m} \right\rfloor}{m} \right\rceil = \left\lfloor \frac{n+m-1-m\left\lfloor \frac{n}{m} \right\rfloor}{m} \right\rfloor$$

We know that

$$(n \bmod m) = n - m\left\lfloor \frac{n}{m} \right\rfloor$$

Therefore, we can perform the following substitutions:

$$\left\lceil \frac{n \bmod m}{m} \right\rceil = \left\lfloor \frac{n \bmod m + m - 1}{m} \right\rfloor$$

Or

$$\left\lceil \frac{n \bmod m}{m} \right\rceil = \left\lfloor \frac{n \bmod m}{m} + \frac{m-1}{m} \right\rfloor$$

Problem 3.12

From the previous slide:

$$\left\lceil \frac{n \bmod m}{m} \right\rceil = \left\lfloor \frac{n \bmod m}{m} + \frac{m-1}{m} \right\rfloor$$

To prove the original statement, it is enough to prove the above expression.

Lets break the statement into right-hand side (RHS) and left-hand side (LHS):

$$LHS = \left\lceil \frac{n \bmod m}{m} \right\rceil \qquad RHS = \left\lfloor \frac{n \bmod m}{m} + \frac{m-1}{m} \right\rfloor$$

We will consider 2 cases:

- **$m = 1$**
- **$m > 1$**

Case 1: m = 1

$LHS = \lceil \frac{n \bmod m}{m} \rceil$ $\quad$ $RHS = \lfloor \frac{n \bmod m}{m} + \frac{m-1}{m} \rfloor$

$$LHS = \lceil \frac{n \bmod m}{m} \rceil = \lceil \frac{n - m\lfloor \frac{n}{m} \rfloor}{m} \rceil = \lceil n - \lfloor n \rfloor \rceil = 0$$

(since n is an integer and m = 1)

$$RHS = \lfloor \frac{n \bmod m}{m} + \frac{m-1}{m} \rfloor = \lfloor \frac{n \bmod m}{m} + 0 \rfloor = \lfloor \frac{n - m\lfloor \frac{n}{m} \rfloor}{m} \rfloor = \lfloor n - \lfloor n \rfloor \rfloor = 0$$

(since n is an integer, m = 1 and $\frac{m-1}{m} = 0$)

LHS = RHS

Thus

$$\lceil \frac{n \bmod m}{m} \rceil = \lfloor \frac{n \bmod m}{m} + \frac{m-1}{m} \rfloor$$

Case 2: m > 1

$LHS = \left\lceil \frac{n \bmod m}{m} \right\rceil$ $RHS = \left\lfloor \frac{n \bmod m}{m} + \frac{m-1}{m} \right\rfloor$

Case 2.1: **(n mod m) = 0**

$LHS = 0$

$RHS = \left\lfloor 0 + \frac{m-1}{m} \right\rfloor = 0$

$LHS = RHS$

Thus

$$\left\lceil \frac{n \bmod m}{m} \right\rceil = \left\lfloor \frac{n \bmod m}{m} + \frac{m-1}{m} \right\rfloor$$

Case 2.2: **0 < (n mod m) < m**

OR

$1 \le (n \bmod m) \le (m - 1)$

Continued on the next slide…

Case 2: m > 1

LHS = $\left\lceil \frac{n \bmod m}{m} \right\rceil$ RHS = $\left\lfloor \frac{n \bmod m}{m} + \frac{m-1}{m} \right\rfloor$

Case 2.2: $1 \le (n \bmod m) \le (m - 1)$

$$1 \le (n \bmod m) \le (m - 1)$$
$$1/m \le (n \bmod m)/m \le (m - 1)/m$$
$$1/m + (m - 1)/m \le (n \bmod m)/m + (m-1)/m \le (m - 1)/m + (m - 1)/m$$
$$1 \le (n \bmod m)/m + (m-1)/m \le (2m - 2)/m$$
$$(2m - 2)/m = (2 - 2/m) < 2, \text{ where } m \ge 2$$
$$1 \le (n \bmod m)/m + (m-1)/m < 2$$

Considering the above:

LHS = $\left\lceil \frac{n \bmod m}{m} \right\rceil = 1$ RHS = $\left\lfloor \frac{n \bmod m}{m} + \frac{m-1}{m} \right\rfloor = 1$

Thus

$$\left\lceil \frac{n \bmod m}{m} \right\rceil = \left\lfloor \frac{n \bmod m}{m} + \frac{m-1}{m} \right\rfloor$$

Problem 3.12: Conclusions

We proved that

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor$$

for all integers n and all positive integers m.