cse547

- Chapter 2, Problems 9, 10

Chapter 2 Problem 9

- What is the law of exponents for rising factorial power, analogous to (2.52)

$$x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}$$

Use this to define $x^{\overline{-n}}$

Observation

$$x^{\overline{4}} = x(x+1)(x+2)(x+3)$$

$$x^{\overline{3}} = x(x+1)(x+2)$$

$$x^{\overline{2}} = x(x+1)$$

$$x^{\overline{1}} = x$$

$$x^{\overline{0}} = 1$$

- $x^{\overline{m}}$ and $x^{\overline{m+1}}$ are similar, only differ in one factor.

- Conjecture, for integers m,n>=0

$$x^{\overline{m+1}} = x^{\overline{m}}(x+m)$$

- More general version, for integers m,n>=0

$$x^{\overline{m+n}} = x^{\overline{m}}(x+m)^{\overline{n}}$$

Conjecture Prove

- By definition, for integers m,n>=0

$$x^{\overline{m}} = x(x+1)(x+2)...(x+m-1)$$

$$x^{\overline{m+n}} = x(x+1)...(x+m-1)(x+m)(x+m+1)...(x+m+n-1)$$

- Plug in the same factor

$$x^{\overline{m+n}} = x^{\overline{m}}(x+m)(x+m+1)...(x+m+n-2)(x+m+n-1)$$

- Since,

$$(x+m)^{\overline{n}} = (x+m)(x+m+1)...(x+m+n-1)$$

we get $x^{\overline{m+n}} = x^{\overline{m}}(x+m)^{\overline{n}}$, for integers m,n>=0

Deduction

- For integers m, n>=0, and m>=n

$$x^{\overline{m-n}} = x(x+1)\ldots(x+m-n-1)$$

$$x^{\overline{m}} = x^{\overline{m-n+n}} = x^{\overline{m-n}}(x+m-n)^{\overline{n}}$$

$$x^{\overline{m-n}} = x^{\overline{m}} / (x+m-n)^{\overline{n}}$$

- Then,
- If we let m=0

$$x^{\overline{-n}} = x^{\overline{0}} / (x-n)^{\overline{n}} = 1/(x-n)^{\overline{n}}$$

Definition

- Define, for integer n>=0

$$x^{\overline{-n}} = \frac{1}{(x-n)^{\overline{n}}} = \frac{1}{(x-n)(x-n+1)...(x-1)}$$

- By the definition above, we shall prove $x^{\overline{m+n}} = x^{\overline{m}}(x+m)^{\overline{n}}$, for integers m, n

Proof

- Case 1: integers m, n>=0 is proved.
- Case 2: integers m, n<0
  - By the definition of $x^{\overline{n}}$ for integer n<0

$$x^{\overline{m+n}} = 1/(x+m+n)^{\overline{-(m+n)}}$$

  - Since, −m>0 and −n>0

$$x^{\overline{m+n}} = \frac{1}{(x+m+n)^{\overline{-n}}(x+m+n-n)^{\overline{-m}}}$$

$$= \frac{1}{(x+m)^{\overline{-m}}} \cdot \frac{1}{(x+m+n)^{\overline{-n}}}$$

$$= x^{\overline{m}}(x+m)^{\overline{n}}$$

Proof continue

- Case 3: integers m>0 and n<0
  - If (m+n)>=0

$$x^{\overline{m+n}} = x(x+1)\ldots(x+m+n-1)$$

$$= \frac{x(x+1)\ldots(x+m+n-1)(x+m+n)\ldots(x+m-1)}{(x+m+n)(x+m+n+1)\ldots(x+m-1)}$$

$$= \frac{x^{\overline{m}}}{(x+m+n)^{\overline{-n}}}$$

$$= x^{\overline{m}}(x+m)^{\overline{n}}$$

Case 3 continue

- If (m+n)<0

$$x^{\overline{m+n}} = \frac{1}{(x+m+n)^{\overline{-(m+n)}}}$$

$$= \frac{1}{(x+m+n)(x+m+n+1)\ldots(x-1)}$$

$$= \frac{x(x+1)\ldots(x+m-1)}{(x+m+n)(x+m+n+1)\ldots(x-1)x\ldots(x+m-1)}$$

$$= \frac{x^{\overline{m}}}{(x+m+n)^{\overline{-n}}}$$

$$= x^{\overline{m}}(x+m)^{\overline{n}}$$

Proof continue

- Case 4: integers m<0 and n>0
  - If (m+n)>=0

$$x^{\overline{m+n}} = x(x+1)\dots(x+m+n-1)$$

$$= \frac{(x+m)(x+m+1)\dots x(x+1)\dots(x+m+n+1)}{(x+m)(x+m+1)\dots(x-1)}$$

$$= \frac{(x+m)^{\overline{n}}}{(x+m)^{\overline{-m}}}$$

$$= x^{\overline{m}}(x+m)^{\overline{n}}$$

Case 4 continue

- If (m+n)<0

$$x^{\overline{m+n}} = \frac{1}{(x+m+n)^{\overline{-(m+n)}}}$$

$$= \frac{1}{(x+m+n)(x+m+n+1)\ldots(x-1)}$$

$$= \frac{(x+m)(x+m+1)\ldots(x+m+n-1)}{(x+m)(x+m+1)\ldots(x+m+n)(x+m+n+1)\ldots(x-1)}$$

$$= \frac{(x+m)^{\overline{n}}}{(x+m)^{\overline{-m}}}$$

$$= x^{\overline{m}}(x+m)^{\overline{n}}$$

Conclusion for P9

- We find the law of exponents for rising factorial power $x^{\overline{m+n}} = x^{\overline{m}}(x+m)^{\overline{n}}$
- We define a good expression of $x^{\overline{-n}}$
- We prove that by our definition, the law holds for integers m, n

Chapter 2 Problem 10

- The text derives the following formula for the difference of a product:

$$\Delta(uv) = u\Delta v + Ev\Delta u$$

How can this formula be correct, when the left-hand side is symmetric with respect to u and v but the right-hand side is not

In order to solve this problem, we have to realize how this formula is derived from text book: P55

$$
\begin{aligned}
\Delta(u(x)v(x)) &= u(x+1)v(x+1) - u(x)v(x) \\
&= u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x) \\
&= u(x)\Delta v(x) + v(x+1)\Delta u(x)
\end{aligned}
$$

By using a shift operator E, defined by Ef(x)=f(x+1)

We get

$$\Delta(uv) = u\Delta v + Ev\Delta u$$

Where is this non-symmetric from?

step1:

$$\Delta(u(x)v(x)) = u(x+1)v(x+1) - u(x)v(x)$$

symmetric:

step2:

$$\Delta(u(x)v(x)) = u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x)$$

non-symmetric:

step3:

$$\Delta(u(x)v(x)) = u(x)\Delta v(x) + v(x+1)\Delta u(x)$$

non-Symmetric:

We get the symbol E because we add and minus a same element u(x)v(x+1)

$$\begin{aligned}\Delta(u(x)v(x)) &= u(x+1)v(x+1) - u(x)v(x) \\ &= u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x) \\ &= u(x)\Delta v(x) + v(x+1)\Delta u(x)\end{aligned}$$

What if we add and minus a same element u(x+1)v(x)?

$$\begin{aligned}\Delta(u(x)v(x)) &= u(x+1)v(x+1) - u(x)v(x) \\ &= u(x+1)v(x+1) - u(x+1)v(x) + u(x+1)v(x) - u(x)v(x) \\ &= v(x)\Delta u(x) + u(x+1)\Delta v(x)\end{aligned}$$

Still not symmetric? Add them together!

$$
\begin{aligned}
2\Delta(u(x)v(x)) &= 2(u(x+1)v(x+1) - u(x)v(x)) \\
&= u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x) \\
&\quad + u(x+1)v(x+1) - u(x+1)v(x) + u(x+1)v(x) - u(x)v(x) \\
&= u(x)\Delta v(x) + v(x+1)\Delta u(x) + u(x+1)\Delta v(x) + v(x)\Delta u(x)
\end{aligned}
$$

These steps are all symmetric!

Then we can use symbol E, we get:

$$2\Delta(uv) = u\Delta v + Ev\Delta u + Eu\Delta v + v\Delta u$$

This is also symmetric!

We have a symmetric form!

So the form should be:

$$2\Delta(uv) = u\Delta v + Ev\Delta u + Eu\Delta v + v\Delta u$$

In fact, if we separate the above form ,we get two forms which are symmetric to each other:

$$\Delta(uv) = u\Delta v + Ev\Delta u$$

$$\Delta(uv) = v\Delta u + Eu\Delta v$$

Why we have symmetric form in the continuous case?

We still have doubt in it.

Compare with the continuous case

1.Recall the derivative for continuous function

$$(u(x)v(x))' = \lim_{h->0} \frac{u(x+h)v(x+h)-u(x)v(x)}{h}$$

$$= \lim_{h->0} \frac{u(x+h)v(x+h)-u(x)v(x+h)+u(x)v(x+h)-u(x)v(x)}{h}$$

$$= \lim_{h->0} \frac{u(x+h)-u(x)}{h} v(x+h) + \lim_{h->0} \frac{v(x+h)-v(x)}{h} u(x)$$

$$= \lim_{h->0} \frac{u(x+h)-u(x)}{h} \lim_{h->0} v(x+h) + \lim_{h->0} \frac{v(x+h)-v(x)}{h} u(x)$$

$$= u'(x) \lim_{h->0} v(x+h) + u(x)v(x)'$$

$$= u'(x)v(x) + u(x)v(x)'$$

2.discrete case:

$$\Delta uv = u(x)\Delta v(x) + v(x+1)\Delta u(x)$$

Conclusion of P10

1. This formula is correct.

2. The right-hand side seems non-symmetric, but in fact it is symmetric of u and v in the discrete case.

3. For continuous function, it is symmetric.