CSE547
Chapter 2 Problems 27, 29

Problem 27

Question : Compute $\Delta(c^x)$, and use it to deduce the value of

$$\sum_{k=1}^{n}((-2)^k / k)$$

Computation of $\Delta(c^{\underline{x}})$

We know that (from class notes) ,

$\Delta f(x) = f(x+1) - f(x)$ and

$$x^{\underline{m}} = \underbrace{x(x-1)(x-2)\ldots(x-m+1)}_{\text{m factors}}, \text{ integer } m \geq 0 \quad (2.43)$$

Computation of $\Delta(c^{\underline{x}})$

Using the formulae given in the previous slide,

$$\Delta(c^{\underline{x}}) = c^{\underline{x+1}} - c^{\underline{x}} \qquad (\text{ Equation I })$$

$$c^{\underline{x+1}} = \underbrace{c(c-1)(c-2)\ \ldots\ (c-x)}_{(x+1)\text{ factors}}$$

$$c^{\underline{x}} = \underbrace{c(c-1)(c-2)\ \ldots\ (c-x+1)}_{x\text{ factors}}$$

Computation of $\Delta(c^{\underline{x}})$

Substituting values of $c^{\underline{x+1}}$ and $c^{\underline{x}}$ in equation(I) we get,

$\Delta(c^{\underline{x}}) = (\mathbf{c(c-1)(c-2)\ldots(c-x+1)}(c-x)) - (\mathbf{c(c-1)(c-2)\ldots(c-x+1)})$

Taking **c(c-1)(c-2) … (c-x+1)** common,

$$\begin{aligned}\Delta(c^{\underline{x}}) &= c(c-1)(c-2)\ldots(c-x+1)\ ((c-x) -1) \\ &= c(c-1)(c-2)\ldots(c-x+1)\ \ (c-x-1) \qquad \text{(Equation II)}\end{aligned}$$

↑

(c-x) term is missing for $c^{\underline{x+2}}$

Computation of $\Delta(c^{\underline{x}})$

Hence, multiplying and dividing equation(II) by (c-x) we get,

$$\Delta(c^{\underline{x}}) = \frac{c(c-1)(c-2)\ldots(c-x+1)\,\mathbf{(c-x)}(c-x-1)}{\mathbf{(c-x)}}$$

By definition,

$$c^{\underline{x+2}} = c(c-1)(c-2)\ldots(c-x-1)$$

$$\therefore \quad \Delta(c^{\underline{x}}) = c^{\underline{x+2}}/(c-x) \qquad \text{(Equation III)}$$

Deducing value of $\sum_{k=1}^{n}((-2)^{k}/k)$

In order to deduce the value of $\sum_{k=1}^{n}((-2)^{k}/k)$ using the calculated value of $\Delta(c^{x})$,

We substitute c = -2 and x = x-2 in equation(II)

$$\Delta((-2)^{x-2}) = \frac{(-2)^{(x-2)+2}}{((-2)-(x-2))}$$

$$= \frac{(-2)^{x}}{-x} \qquad \text{(Equation IV)}$$

Deducing value of $\sum_{k=1}^{n}\left((-2)^{k}/k\right)$

<u>Extra Fact:</u>

$$
\begin{aligned}
-\Delta f(x) &= -(f(x+1) - f(x)) \\
&= f(x) - f(x+1) \\
&= -f(x+1) - (-f(x)) \\
&= \Delta(-f(x))
\end{aligned}
$$

$$\therefore \quad \Delta\left(-(-2)^{x-2} \right) = \frac{(-2)^{x}}{x} \qquad \text{(Equation V)}$$

Deducing value of $\sum_{k=1}^{n}((-2)^{k}/k)$

$$\sum_{a}^{b} g(x)\delta x = \sum_{k=a}^{b-1} g(k) \quad \text{for integers } b \geq a \qquad (2.48)$$

Since, $\sum_{k=1}^{n}((-2)^{k}/k)$ is of the form

$\sum_{k=a}^{b-1} g(k)$ we get,

$$\sum_{k=1}^{n}((-2)^{k}/k) = \sum_{k=1}^{n+1} \frac{(-2)^{k}}{k} \delta k, \quad \text{for } n \geq 0 \qquad (\text{ Equation VI })$$

We know that,

$$g(x) = \Delta f(x) \quad \text{iff} \quad \sum g(x)\,\delta x = f(x) + c \qquad (2.46)$$

Deducing value of $\sum_{k=1}^{n}((-2)^{\underline{k}}/k)$

Using Equations IV, VI and 2.46, we get

$$\sum_{k=1}^{n+1} \frac{(-2)^{\underline{k}}}{k}\,\delta k = -(-2)^{\underline{k-2}}\Big|_{1}^{n+1}$$

$$= -(-2)^{\underline{n+1-2}} - (-(-2)^{\underline{1-2}})$$

$$= -(-2)^{\underline{n-1}} - (-(-2)^{\underline{-1}})$$

$$= -(-2)^{\underline{n-1}} + (-2)^{\underline{-1}}$$

$$= (-2)^{\underline{-1}} - (-2)^{\underline{n-1}} \qquad \text{(Equation VII)}$$

Deducing value of $\sum_{k=1}^{n}\left((-2)^{\underline{k}}/k\right)$

We know that,

$$x^{\underline{-m}} = \frac{1}{(x+1)(x+2)\dots(x+m)} \quad \text{for } m>0 \qquad (2.51)$$

$$\therefore \quad x^{\underline{-1}} = 1/(x+1)$$

$$\Rightarrow (-2)^{\underline{-1}} = 1/(-2+1)$$

$$= 1/(-1)$$

$$= -1$$

Deducing value of $\sum_{k=1}^{n}((-2)^{\underline{k}}/k)$

Substituting in equation (VII) we get,

$$\sum_{k=1}^{n+1} \frac{(-2)^{\underline{k}}}{k}\,\delta k = -1 - (-2)^{\underline{n-1}}$$

$$= -1 - ((-2)(-2-1)(-2-2)\ldots(-2-(n-2)))$$

$$= -1 - ((-2)(-3)(-4)\ldots(-n))$$

$$= -1 + ((-1)(-2)(-3)\ldots(-n))$$

$$= -1 + (-1)^{n}\, n!$$

Verification

$$\sum_{k=1}^{n}((-2)^k / k) \quad = \quad -1 + (-1)^n \, n!$$

n	1	2	3	4
$\sum_{k=1}^{n}((-2)^k/k)$	-2	1	-7	23
$-1 + (-1)^n\, n!$	-2	1	-7	23

Question 29 of chapter 2

Evaluate $$\sum_{k=1}^{n} \frac{(-1)^k k}{(4k^2 - 1)}$$

Evaluating $\sum_{k=1}^{n} \frac{(-1)^k k}{(4k^2 - 1)}$

$$\frac{k}{4k^2 -1} = \frac{k}{(2k)^2 - (1)^2}$$

$$= \frac{k}{(2k -1)(2k +1)}$$

Using partial fractions,

$$\frac{k}{(2k -1)(2k+1)} = \frac{A}{(2k-1)} + \frac{B}{(2k+1)} \quad (\text{ eqn I })$$

Evaluating $\sum_{k=1}^{n} \frac{(-1)^k k}{(4k^2 - 1)}$

Multiplying and dividing by (2k -1) * (2k +1)

we get

$$k = A\,(2k + 1) + B\,(2k - 1)$$

Grouping powers of k,

$$k + 0 = ((2A)\,k) + A + ((2B)\,k) - B$$

Equating powers of k on both sides, we get the linear equations:

$2A + 2B = 1$ and (equation 1)

$A - B = 0$ (equation 2)

Evaluating $\sum_{k=1}^{n} \frac{(-1)^k k}{(4k^2 - 1)}$

Solving the simultaneous equations obtained in the previous slide , we get

2 A + 2 B = 1

2A - 2 B = 0 (multiplying equation 2 by 2)

4A = 1 ➔ A = 1/4

From equation 2, B = 1/4

Evaluating $\sum_{k=1}^{n} \frac{(-1)^k k}{(4k^2 - 1)}$

$$\therefore \qquad \frac{k}{4k^2 - 1} = \frac{1}{4}\left(\frac{1}{(2k-1)} + \frac{1}{(2k+1)} \right)$$

$$\sum_{k=1}^{n} \frac{(-1)^k k}{(4k^2-1)} = \sum_{k=1}^{n} (-1)^k \frac{1}{4}\left(\frac{1}{(2k-1)} + \frac{1}{(2k+1)} \right)$$

Evaluating $\sum_{k=1}^{n} \frac{(-1)^k k}{(4k^2 - 1)}$

We can split the sum on the right side into two summations as follows :

$$\sum_{k=1}^{n} \frac{(-1)^k 1}{4} \left(\frac{1}{(2k -1)} \right) + \sum_{k=1}^{n} \frac{(-1)^k 1}{4} \left(\frac{1}{(2k +1)} \right)$$

This can be changed to a harmonic sum by putting 2k-1 = m and 2k +1 = m but that would make it complex.

Evaluating $\sum_{k=1}^{n} \frac{(-1)^k k}{(4k^2 - 1)}$

Expanding the summation we get

$$\frac{(-1)^1 1}{4}\left(\frac{1}{1} + \frac{1}{3}\right) + \frac{(-1)^2 1}{4}\left(\frac{1}{3} + \frac{1}{5}\right)$$

$$+ \cdots$$

$$+ \frac{(-1)^n 1}{4}\left(\frac{1}{(2n - 1)} + \frac{1}{(2n + 1)}\right)$$

Evaluating $\sum_{k=1}^{n} \frac{(-1)^k k}{(4k^2 - 1)}$

Expanding the summation we get

$$\frac{1}{4}\left(\frac{(-1)}{1} + \frac{(-1)}{3}\right) + \frac{1}{4}\left(\frac{1}{3} + \frac{1}{5}\right)$$

$$+ \quad \frac{1}{4}\left(\frac{(-1)}{5} + \frac{(-1)}{7}\right) + \cdots +$$

$$\frac{(-1)^n 1}{4}\left(\frac{1}{(2n - 1)} + \frac{1}{(2n + 1)}\right)$$

Evaluating $\sum_{k=1}^{n} \frac{(-1)^k k}{(4k^2 - 1)}$

We can see that the alternate terms get cancelled leaving the first and the last term alone.

$$\frac{-1}{4} + \frac{(-1)^n}{4} \left(\frac{1}{(2n + 1)} \right)$$

Which is the required solution.

Verification

$$\sum_{k=1}^{n} \frac{(-1)^k k}{(4k^2 - 1)} = \frac{-1}{4} + \frac{(-1)^n}{4}\left(\frac{1}{(2n + 1)}\right)$$

n	1	2	3	4
$\sum_{k=1}^{n}(((-1)^k k)/(4k^2-1))$	-1/3	-1/5	-2/7	-2/9
$(-1/4)+(((1)^n/4)(1/(2n+1))$	-1/3	-1/5	-2/7	-2/9