CSE547
chapter 2
Problems 20,21

Chap.2 Exercise 20

Try to evaluate $\sum_{0 \le k \le n} kH_k$ by the perturbation method, but deduce the value of $\sum_{0 \le k \le n} H_k$ instead.

Perturbation Method

$$S_n = \sum_{0 \le k \le n} a_k$$

$$S_n + a_{n+1} = \sum_{0 \le k \le n+1} a_k = a_0 + \sum_{1 \le k \le n+1} a_k$$

$$= a_0 + \sum_{1 \le k+1 \le n+1} a_{k+1}$$

$$= a_0 + \sum_{0 \le k \le n} a_{k+1}$$

Harmonic Number

$$H_n = 1 + \frac{1}{2} +, \ldots, + \frac{1}{n} = \sum_{1 \le k \le n} \frac{1}{k}$$

Ex 20) $$S_n = \sum_{0 \le k \le n} kH_k$$

$$S_n + (n+1)H_{n+1} = \sum_{0 \le k \le n+1} kH_k = 0H_0 + \sum_{1 \le k \le n+1} kH_k$$

$$= 0 + \sum_{1 \le k+1 \le n+1} (k+1)H_{k+1}$$

$$= \sum_{0 \le k \le n} (k+1)H_{k+1}$$

$$H_n = \sum_{1 \le k \le n} \frac{1}{k}$$

$$H_{n+1} = \sum_{1 \le k \le n} \frac{1}{k} + \frac{1}{n+1}$$

$$S_n + (n+1)H_{n+1} = \sum_{0 \le k \le n} (k+1)H_{k+1}$$

$$= \sum_{0 \le k \le n} (k+1)(\frac{1}{k+1} + H_k)$$

$$= \sum_{0 \le k \le n} 1 + \sum_{0 \le k \le n} kH_k + \sum_{0 \le k \le n} H_k$$

$$S_n + (n+1)H_{n+1} = \sum_{0 \le k \le n} 1 + \sum_{0 \le k \le n} kH_k + \sum_{0 \le k \le n} H_k$$

$$(n+1)H_{n+1} = \sum_{0 \le k \le n} 1 + \sum_{0 \le k \le n} H_k$$

$$\sum_{0 \le k \le n} H_k = (n+1)H_{n+1} - \sum_{0 \le k \le n} 1$$

$$\sum_{0 \le k \le n} H_k = (n+1)H_{n+1} - (n+1) \qquad //End\ of\ ex.20$$

Perturbation Method

Excercises 21

Evaluate by perturbation method

$$S_n = \sum_{k=0}^{n} (-1)^{n-k}, \quad T_n = \sum_{k=0}^{n} (-1)^{n-k} k,$$

and $U_n = \sum_{k=0}^{n} (-1)^{n-k} k^2$

Assume $n \geq 0$

$$S_n = \sum_{0\le k\le n} (-1)^{n-k}$$

Which implies,

$$S_{n+1} = \sum_{0\le k\le n+1} (-1)^{(n+1)-k}$$

Split off the first term

$$S_{n+1} = \sum_{0\le k\le n+1} (-1)^{(n+1)-k}$$

$$= a_0 + \sum_{1\le k\le n+1} (-1)^{n+1-k}$$

$$= (-1)^{n+1-0} + \sum_{1\le k+1\le n+1} (-1)^{(n+1)-(k+1)}$$

$$= (-1)^{n+1} + \sum_{0\le k\le n} (-1)^{n-k}$$

$$= (-1)^{n+1} + S_n \ldots\ldots\ldots\ldots\ldots (a1)$$

$$S_n = \sum_{0\le k\le n} (-1)^{n-k}$$

Split off the final term

$$S_{n+1} = \sum_{0\le k\le n+1} (-1)^{(n+1)-k}$$

$$= \sum_{0\le k\le n} (-1)^{n+1-k} + (-1)^{(n+1)-(n+1)}$$

$$= \sum_{0\le k\le n} (-1)^{n+1-k} + 1$$

$$= -\sum_{0\le k\le n} (-1)^{n-k} + 1$$

$$= 1 - S_n \dots\dots\dots\dots\dots\dots\dots (a2)$$

Equating (a1) and (a2) we get….

$$1 - S_n = (-1)^{n+1} + S_n$$

$$\Rightarrow 2S_n = 1 - (-1)^{n+1}$$

$$\Rightarrow S_n = \frac{1 + (-1)^n}{2}$$

$$T_n = \sum_{k=0}^{n} (-1)^{n-k} k$$

Which implies

$$T_{n+1} = \sum_{k=0}^{n+1} (-1)^{n+1-k} k$$

Split off the first term

$$T_{n+1} = \sum_{k=0}^{n+1} (-1)^{n+1-k} k$$

$$= 0.(-1)^{(n+1)} + \sum_{k=1}^{n+1} (-1)^{n+1-k} k$$

$$= \sum_{1 \le k+1 \le n+1} (-1)^{n+1-(k+1)} (k+1)$$

$$= \sum_{0 \le k \le n} (-1)^{n-k} (k+1)$$

$$= \sum_{0 \le k \le n} (-1)^{n-k} k + \sum_{0 \le k \le n} (-1)^{n-k}$$

$$= T_n + S_n \ldots\ldots\ldots\ldots\ldots (b1)$$

Split off the final term

$$T_{n+1} = \sum_{k=0}^{n+1} \left(-1\right)^{n+1-k} k$$

$$= \sum_{k=0}^{n} \left(-1\right)^{n+1-k} k + \left(-1\right)^{n+1-(n+1)} (n+1)$$

$$= -\sum_{k=0}^{n} \left(-1\right)^{n-k} k + (n+1)$$

$$= (n+1) - T_n \text{.......................}(b2)$$

Equating (b1) and (b2)……

$$(n+1) - T_n = T_n + S_n$$

$$\Rightarrow 2T_n = (n+1) - (1 + (-1)^n)$$

$$\Rightarrow T_n = \frac{1}{2}(n - (-1)^n)$$

$$U_n = \sum_{k=0}^{n} (-1)^{n-k} k^2$$

Which implies

$$U_{n+1} = \sum_{k=0}^{n+1} (-1)^{n+1-k} k^2$$

Split off the first term

$$U_{n+1} = 0.(-1)^{(n+1)-0} + \sum_{k=1}^{n+1} (-1)^{n+1-k} k^2$$

$$= \sum_{k+1=1}^{n+1} (-1)^{n+1-(k+1)} (k+1)^2$$

$$= \sum_{k=0}^{n} (-1)^{n-k} (k^2 + 2k + 1)$$

$$= \sum_{k=0}^{n} (-1)^{n-k} k^2 + \sum_{k=0}^{n} (-1)^{n-k} 2k + \sum_{k=0}^{n} (-1)^{n-k}$$

$$= U_n + 2T_n + S_n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (c1)$$

Split off the final term

$$U_{n+1} = \sum_{k=0}^{n} (-1)^{n+1-k} k^2 + (-1)^{n+1-(n+1)} (n+1)^2$$

$$= -\sum_{k=0}^{n} (-1)^{n-k} k^2 + (n+1)^2$$

$$= -U_n + (n+1)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (c2)$$

Equating (c1) and (c2)……

$$(n+1)^2 - U_n = U_n + 2T_n + S_n$$

$$\Rightarrow 2U_n = (n+1)^2 - 2T_n - S_n$$

$$\Rightarrow 2U_n = (n+1)^2 - (n - (-1)^n) - (1 + (-1)^n)$$

$$\Rightarrow U_n = \frac{n^2 + n}{2}$$