CSE547

Chapter 1, Problem 8

Problem 8

- Solve the recurrence
- $Q_n \neq 0 \;;\; Q_0 = \alpha \;;\; Q_1 = \beta \;;$
- $Q_n = (1 + Q_{n-1}) / Q_{n-2}, \text{ for n} > 1$
- Assume that $Q_n \neq 0 \text{ for all n} \geq 0$

Observation

$$Q_0 = \alpha \qquad Q_1 = \beta \qquad Q_n \neq 0 \qquad Q_n = (1 + Q_{n-1}) / Q_{n-2}, \text{ for n} > 1$$

Find each term by substitution

$$Q_2 = \frac{(1 + Q_1)}{Q_0} = \frac{(1 + \beta)}{\alpha}$$

$$Q_3 = \frac{(1 + Q_2)}{Q_1} = \frac{(1 + \frac{1 + \beta}{\alpha})}{\beta} = \frac{1 + \alpha + \beta}{\frac{\alpha}{\beta}} = \frac{\alpha + \beta + 1}{\alpha\beta}$$

Observation (Cont.)

$$Q_4 = \frac{(1+Q_3)}{Q_2} = \frac{1+\dfrac{\alpha+\beta+1}{\alpha\beta}}{\dfrac{1+\beta}{\alpha}} = \frac{\dfrac{\alpha\beta+\alpha+\beta+1}{\alpha\beta}}{\dfrac{1+\beta}{\alpha}}$$

$$= \left(\frac{\alpha\beta+\alpha+\beta+1}{\alpha\beta}\right)\times\left(\frac{\alpha}{1+\beta}\right) = \frac{\alpha\beta+\alpha+\beta+1}{\beta(1+\beta)}$$

$$= \frac{\alpha(\beta+1)+(\beta+1)}{\beta(1+\beta)} = \left(\frac{1+\alpha}{\beta}\right)$$

Observation (Cont.)

$$Q_5 = \frac{1+Q_4}{Q_3} = \frac{1+\frac{(1+\alpha)}{\beta}}{\frac{\alpha+\beta+1}{\alpha\beta}} = \frac{1+\alpha+\beta}{\beta} \times \frac{\alpha\beta}{\alpha+\beta+1}$$

$$= \alpha = Q_0$$

$$Q_6 = \frac{1+Q_5}{Q_4} = \frac{1+\alpha}{\frac{(1+\alpha)}{\beta}} = \beta = Q_1$$

Observation (Cont.)

$$Q_7 = \frac{1+Q_6}{Q_5} = \frac{1+\beta}{\alpha} = Q_2$$

This is Cyclic!!

Mathematical Induction

- Assumption

- P(k) for all k<n+1 prove P(k+1)
- When n = 5k (k >= 0)

$$Q_n = \alpha$$

- When n = 5k + 1

$$Q_n = \beta$$

Assumption (cont.)

- When n = 5k + 2

$$Q_n = \frac{1+\beta}{\alpha}$$

- When n = 5k + 3

$$Q_n = \frac{\alpha+\beta+1}{\alpha\beta}$$

- When n = 5k + 4

$$Q_n = \frac{1+\alpha}{\beta}$$

Mathematical Induction Proof

- When k = 0

$$Q_0 = \alpha \qquad Q_1 = \beta \qquad Q_2 = \frac{1+\beta}{\alpha}$$

$$Q_3 = \frac{\alpha + \beta + 1}{\alpha\beta} \qquad Q_4 = \frac{1+\alpha}{\beta}$$

fine when k = 0.

Induction Step

- When k = m

$$Q_{5m} = \alpha \quad Q_{5m+1} = \beta \quad Q_{5m+2} = \frac{1+\beta}{\alpha}$$

$$Q_{5m+3} = \frac{\alpha+\beta+1}{\alpha\beta} \qquad Q_{5m+4} = \frac{1+\alpha}{\beta}$$

(assumptions)

Induction Step (cont.)

- When k = m + 1

$$Q_{5(m+1)} = \frac{1+Q_{5(m+1)-1}}{Q_{5(m+1)-2}} = \frac{1+Q_{5m+4}}{Q_{5m+3}} = \frac{1+\dfrac{(1+\alpha)}{\beta}}{\dfrac{\alpha+\beta+1}{\alpha\beta}}$$

$$= \alpha$$

Induction Step (cont.)

$$Q_{5(m+1)+1} = \frac{1+Q_{5(m+1)+1-1}}{Q_{5(m+1)+1-2}} = \frac{1+Q_{5(m+1)}}{Q_{5m+4}} = \frac{1+\alpha}{\frac{1+\alpha}{\beta}} = \beta$$

$$Q_{5(m+1)+2} = \frac{1+Q_{5(m+1)+2-1}}{Q_{5(m+1)+2-2}} = \frac{1+Q_{5(m+1)+1}}{Q_{5(m+1)}} = \frac{1+\beta}{\alpha}$$

Induction Step (cont.)

$$Q_{5(m+1)+3} = \frac{1+Q_{5(m+1)+3-1}}{Q_{5(m+1)+3-2}} = \frac{1+Q_{5(m+1)+2}}{Q_{5(m+1)+1}} = \frac{1+\frac{1+\beta}{\alpha}}{\beta}$$

$$= \frac{1+\alpha+\beta}{\alpha\beta}$$

$$Q_{5(m+1)+4} = \frac{1+Q_{5(m+1)+4-1}}{Q_{5(m+1)+4-2}} = \frac{1+Q_{5(m+1)+3}}{Q_{5(m+1)+2}} = \frac{1+\frac{1+\alpha+\beta}{\alpha\beta}}{\frac{1+\beta}{\alpha}}$$

$$= \frac{1+\alpha}{\beta}$$

- So when k = m+1, the assumptions work.

Note

- Since we have 4 assumptions, we have to apply mathematical induction to each assumption.