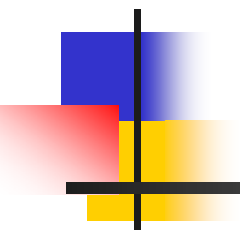


CSE547



Chapter 1, Problem 8



Problem 8

- Solve the recurrence
- $Q_n \neq 0$; $Q_0 = \alpha$; $Q_1 = \beta$;
- $Q_n = (1 + Q_{n-1}) / Q_{n-2}$, for $n > 1$
- Assume that $Q_n \neq 0$ for all $n \geq 0$



Observation

$$Q_0 = \alpha \quad Q_1 = \beta \quad Q_n \neq 0 \quad Q_n = (1 + Q_{n-1}) / Q_{n-2}, \text{ for } n > 1$$

Find each term by substitution

$$Q_2 = \frac{(1 + Q_1)}{Q_0} = \frac{(1 + \beta)}{\alpha}$$

$$Q_3 = \frac{(1 + Q_2)}{Q_1} = \frac{(1 + \frac{1 + \beta}{\alpha})}{\beta} = \frac{1 + \alpha + \beta}{\frac{\alpha}{\beta}} = \frac{\alpha + \beta + 1}{\alpha\beta}$$



Observation (Cont.)

$$\begin{aligned} Q_4 &= \frac{(1+Q_3)}{Q_2} = \frac{1 + \frac{\alpha + \beta + 1}{\alpha\beta}}{\frac{1 + \beta}{\alpha}} = \frac{\frac{\alpha\beta + \alpha + \beta + 1}{\alpha\beta}}{\frac{1 + \beta}{\alpha}} \\ &= \left(\frac{\alpha\beta + \alpha + \beta + 1}{\alpha\beta}\right) \times \left(\frac{\alpha}{1 + \beta}\right) = \frac{\alpha\beta + \alpha + \beta + 1}{\beta(1 + \beta)} \\ &= \frac{\alpha(\beta + 1) + (\beta + 1)}{\beta(1 + \beta)} = \left(\frac{1 + \alpha}{\beta}\right) \end{aligned}$$



Observation (Cont.)

$$\begin{aligned} Q_5 &= \frac{1+Q_4}{Q_3} = \frac{1+\frac{(1+\alpha)}{\beta}}{\frac{\alpha+\beta+1}{\alpha\beta}} = \frac{1+\alpha+\beta}{\beta} \times \frac{\alpha\beta}{\alpha+\beta+1} \\ &= \alpha = Q_0 \end{aligned}$$

$$Q_6 = \frac{1+Q_5}{Q_4} = \frac{1+\alpha}{\frac{(1+\alpha)}{\beta}} = \beta = Q_1$$



Observation (Cont.)

$$Q_7 = \frac{1+Q_6}{Q_5} = \frac{1+\beta}{\alpha} = Q_2$$

This is Cyclic!!



Mathematical Induction

- Assumption
- $P(k)$ for all $k < n+1$ prove $P(k+1)$
- When $n = 5k$ ($k \geq 0$)

$$Q_n = \alpha$$

- When $n = 5k + 1$

$$Q_n = \beta$$



Assumption (cont.)

- When $n = 5k + 2$

$$Q_n = \frac{1 + \beta}{\alpha}$$

- When $n = 5k + 3$

$$Q_n = \frac{\alpha + \beta + 1}{\alpha\beta}$$

- When $n = 5k + 4$

$$Q_n = \frac{1 + \alpha}{\beta}$$



Mathematical Induction Proof

- When $k = 0$

$$Q_0 = \alpha \quad Q_1 = \beta \quad Q_2 = \frac{1 + \beta}{\alpha}$$

$$Q_3 = \frac{\alpha + \beta + 1}{\alpha\beta} \quad Q_4 = \frac{1 + \alpha}{\beta}$$

fine when $k = 0$.



Induction Step

- When $k = m$

$$Q_{5m} = \alpha \quad Q_{5m+1} = \beta \quad Q_{5m+2} = \frac{1+\beta}{\alpha}$$

$$Q_{5m+3} = \frac{\alpha + \beta + 1}{\alpha\beta} \quad Q_{5m+4} = \frac{1+\alpha}{\beta}$$

(assumptions)



Induction Step (cont.)

- When $k = m + 1$

$$\begin{aligned} Q_{5(m+1)} &= \frac{1 + Q_{5(m+1)-1}}{Q_{5(m+1)-2}} = \frac{1 + Q_{5m+4}}{Q_{5m+3}} = \frac{1 + \frac{(1+\alpha)}{\beta}}{\frac{\alpha + \beta + 1}{\alpha\beta}} \\ &= \alpha \end{aligned}$$



Induction Step (cont.)

$$Q_{5(m+1)+1} = \frac{1 + Q_{5(m+1)+1-1}}{Q_{5(m+1)+1-2}} = \frac{1 + Q_{5(m+1)}}{Q_{5m+4}} = \frac{1 + \alpha}{\frac{1 + \alpha}{\beta}} = \beta$$

$$Q_{5(m+1)+2} = \frac{1 + Q_{5(m+1)+2-1}}{Q_{5(m+1)+2-2}} = \frac{1 + Q_{5(m+1)+1}}{Q_{5(m+1)}} = \frac{1 + \beta}{\alpha}$$



Induction Step (cont.)

$$\begin{aligned} Q_{5(m+1)+3} &= \frac{1+Q_{5(m+1)+3-1}}{Q_{5(m+1)+3-2}} = \frac{1+Q_{5(m+1)+2}}{Q_{5(m+1)+1}} = \frac{1+\frac{1+\beta}{\alpha}}{\beta} \\ &= \frac{1+\alpha+\beta}{\alpha\beta} \end{aligned}$$

$$\begin{aligned} Q_{5(m+1)+4} &= \frac{1+Q_{5(m+1)+4-1}}{Q_{5(m+1)+4-2}} = \frac{1+Q_{5(m+1)+3}}{Q_{5(m+1)+2}} = \frac{1+\frac{1+\alpha+\beta}{\alpha\beta}}{\frac{1+\beta}{\alpha}} \\ &= \frac{1+\alpha}{\beta} \end{aligned}$$

- So when $k = m+1$, the assumptions work.



Note

- Since we have 4 assumptions, we have to apply mathematical induction to each assumption.