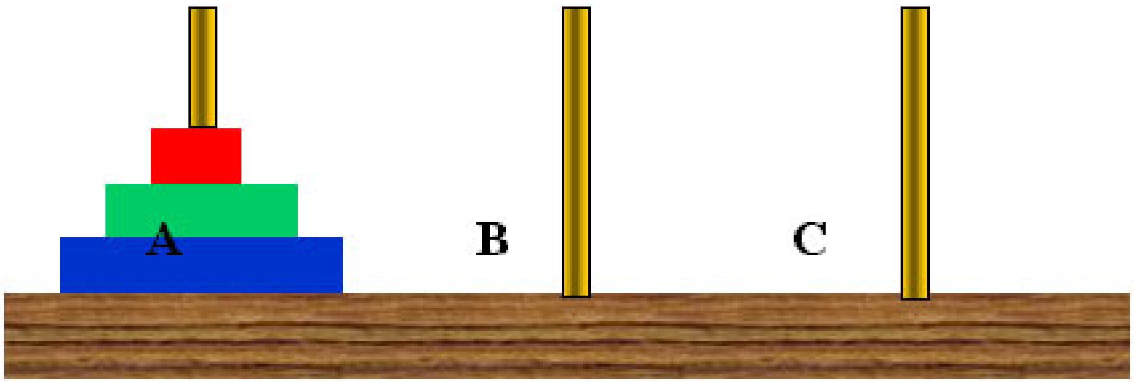
cse547

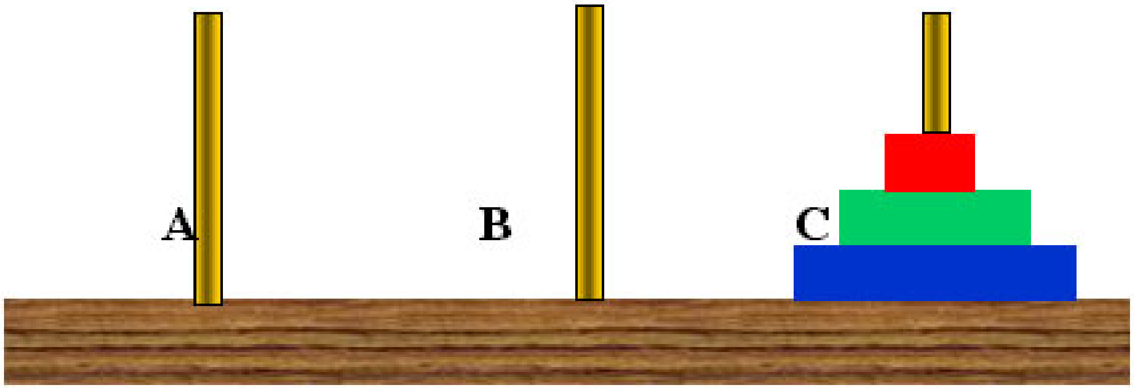
Chapter 1, problem 2

Chapter No 1,Problem No 2

- Question :

  Find the shortest sequence of moves that transfers a tower of n disks from the left peg A to the right peg B, if direct moves between A and B are disallowed. (Each move must be to or from the middle peg. As usual, a larger disk must never appear above a smaller one.)


A
B
C


A
B
C

The objective is to transfer the entire tower from A to C(in the diagram), if direct moves between A and C are disallowed. This problem is a variant of Tower of Hanoi problem.

GENERALIZE :

Lets assume that the tower has “n” disks.

Let T_n be the minimum number of moves that will transfer “n” disks from one peg (i.e. A to another (i.e. C).

Clearly $T_0 = 0$, because no moves at all are needed to transfer a tower of $n = 0$ disks.

T_1 = 2 (Since the peg has to be transferred from A to B and then to C.)

Similarly ,

T_2 = 8

T_3 = 26

Winning Strategy :

(for n = 3)

1. Transfer top 2 disks from A to C (requiring T_2 disk moves).
2. Move the largest disk from A to center peg “B”.
3. Move again the 2 disks from C back to A(requiring T_2 disk moves).

4. Move the largest disk to peg “C”.
5. Again we now need to move 2 disks from A to C (requiring T_2 disk moves)

General Case :

1.Transfer top (n-1) disks from A to C.

2.Move the largest from A to center peg B.

3.Transfer (n-1) disks from C to A back.

4.Move the largest disk to C.

5.Transfer (n-1) disks from A to C again.

Total number of moves =

$$T_{n-1} + 1 + T_{n-1} + 1 + T_{n-1} = 3\ T_{n-1} + 2$$

Recurrence Relation :

$T_0 = 0$

$T_n = 3\ T_{n-1} + 2$

Lets compute successively a few values to guess the closed formula.

$T_0 = 0$

$T_1 = 3*0 + 2 = 2$

$T_2 = 3*2 + 2 = 8$

$T_3 = 3*8 + 2 = 26$

$T_4 = 3*26 + 2 = 80$

Observation :

$T_n = 3^n -1$

Now we have to prove that

Recurrence relation = Closed Formula

Lets apply Mathematical induction :

Recurrence :

$T_0 = 0, T_n = 3 T_{n-1} + 2$

Closed Formula :

$T_n = 3^n - 1 , n >= 0$

Basis case :

$T_0 = 0$

C.F : $T_0 = 3^0 – 1 = 1 – 1 = 0$

Therefore, Recurrence = C.F for n=0

Let us assume that our closed formula is correct for values <= n – 1.

So, now we need to prove : $T_n = 3^n - 1$

Applying the above relation in the recurrence relation, we get

$$T_n = 3\,T_{n-1} + 2$$
$$= 3(3^{n-1} - 1) + 2$$
$$= 3^n - 3 + 2$$
$$= 3^n - 1$$

Hence, the closed formula holds for n as well.

Answer

Therefore,

By Mathematical Induction we proved for all n € N , $T_n = 3^n – 1$