Chapter one

Problem 2 and 14

Problem 2

- Find the shortest sequence of moves that transfers a tower of n disks from the left peg A to right peg B, if direct moves between A and B are disallowed. Each move must be to or from the middle peg. As usual a larger disk must never appear above a smaller one.)

The problem can be solved recursively as follows:

First consider case, n=1, where we have to move a single disk from A to B. Since direct moves are disallowed this requires 2 moves and hence P(1) = 2.

We define the task of moving n disks from peg A to peg B recursively as follows.

- By assumption we know how to move the top n-1 disks from A to B without direct move → P(n-1)
- Move the largest disk from A to the middle → 1
- Again by assumption we know how to move the top n-1 disks from B to A → P(n-1)
- Move the largest disk from the middle to B →1
- Again by assumption we know how to move the top n-1 disks from A to B without direct move → P(n-1)

- After these moves all the n disks will be in order on peg B. Thus we can see that the total moves required to transfer the n disks is $P(n) = 3P(n-1) + 2$.
- We want to guess the close form so we look at small cases: where we know $P(1)=2$;
- $P(2)=3*2 +2 =8$,
- $P(3)=3*8+2=26$,..
- We suggest the solution to this recurrence as: $P(n) =3^n -1$.

Proof by induction for $P(n) = 3^n - 1$:

$P(0) = 3^0 - 1 = 0$

For $n > 0$ we assume that it works

When n is replaced by n-1:

$P(n) = 3P(n-1)+2 = 3(3^{(n-1)} - 1)+2 = 3^n - 1$.

Problem 14

Problem 14

How many pieces of cheese can you obtain from a single thick piece by making five straight slices?
(the cheese must stay in it's original position while you do all the cutting, and **each slice must correspond to a plane in 3D)** Find a recurrence relation for p(n), the maximum number of three dimensional regions that can be defined by n different planes.

We use this problem

- How many slices of pizza can a person obtain by making n straight cuts with pizza knife.

- Which actually is “What is the maximum number L(n) of regions defined by n lines in the plane?”.

  We showed by induction $L(n)=L(n-1)+n$ $n>0$

  And the close formula for that is $L(n) = n(n+1)/2 +1$

- Consider the most general case, where planes inserted are not parallel and no set of more than 2 planes intersect in the same line. For the n’th plane, all the n-1 previously intersected planes will intersect the n’th plane and create n-1 cuts on that plane.
- As it was shown these n-1 cuts will divide the n’th plane into at most 1+ n(n-1)/2 regions.

The most of new pieces by the n'th cut is exactly this number.

Thus the recurrence relation that describe the maximum number of pieces attainable using n cuts is

$$P(n)=P(n-1)+1+n(n-1)/2$$

given that the base case P(0) =1.

P(1) = 2;

P(2) =2+1 +1 = 4 = 1*2*3/6 + 2 + 1

P(3)= 4+3+1 = 8 = 2*3*4/6 + 3 + 1

P(4)= 8+6+1 = 15 = 3*4*5/6 + 4 + 1

P(5)=15+10+1= 26 = 4*5*6/6 + 5 + 1

P(6)=26+15+1 = 42 = 5*6*7/6 + 6 + 1

The solution to this recurrence is

$P(n)=(n-1)n(n+1)/6 +n+1$ which can be proved by induction.

Prove by induction: $P(n)=(n-1)n(n+1)/6 +n+1$

It works for $P(0)=1$;

Assume it holds when n is replaced with n-1 since

$$P(n) = P(n-1) + n(n-1)/2 +1 \text{ then}$$

$$P(n) = (n-2)(n-1)(n)/6 + n +n(n-1)/2 +1$$

And :

$$P(n) = (n-1)n(n+1)/6 + n+1$$