Cse547

- Chapter 1
- Problem 16 GENERALIZATION

Question

Use the Repertoire method to solve the general five-parameter recurrence:

$$g(1) = \alpha$$

$$g(2n+j) = 3g(n) + \varepsilon n^2 + \gamma n + \beta_j,$$

for $j = 0, 1$ and $n \geq 1$ and $n \in N$

Solution

Convert recursive formula into R:

$g(1) = \alpha$
$g(2n + 0) = 3g(n) + \varepsilon n^2 + \gamma n + \beta_0$
$g(2n + 1) = 3g(n) + \varepsilon n^2 + \gamma n + \beta_1$

(for $n \in N$ and $n \geq 1$)

Procedure

- Compute few initial values for R
- Observe values and guess a general format for the required closed form
- Use Repertoire method to find exact closed formula
- If fails at some point, use Binary Expansion

Initial Values for R

$g(1) = \alpha$

$g(2) = g(2(1) + 0) = 3g(1) + \varepsilon + \gamma + \beta_0$

$= 3\alpha + \varepsilon + \gamma + \beta_0$

$g(3) = g(2(1) + 1) = 3g(1) + \varepsilon + \gamma + \beta_1$

$= 3\alpha + \varepsilon + \gamma + \beta_1$

$g(4) = g(2(2) + 0) = 3g(2) + 4\varepsilon + 2\gamma + \beta_0$

$= 9\alpha + 7\varepsilon + 5\gamma + 4\beta_0$

$g(5) = g(2(2) + 1) = 3g(2) + 4\varepsilon + 2\gamma + \beta_1$

$= 9\alpha + 7\varepsilon + 5\gamma + 3\beta_0 + \beta_1$

Initial Values for R

$g(6) = g(2(3) + 0) = 3g(3) + 9\varepsilon + 3\gamma + \beta_0$

$= 9\alpha + 12\varepsilon + 6\gamma + \beta_0 + 3\beta_1$

$g(7) = g(2(3) + 1) = 3g(3) + 9\varepsilon + 3\gamma + \beta_1$

$= 9\alpha + 12\varepsilon + 6\gamma + 4\beta_1$

$g(8) = g(2(4) + 0) = 3g(4) + 16\varepsilon + 4\gamma + \beta_0$

$= 27\alpha + 37\varepsilon + 19\gamma + 13\beta_0$

$g(9) = g(2(4) + 1) = 3g(4) + 16\varepsilon + 4\gamma + \beta_1$

$= 27\alpha + 37\varepsilon + 19\gamma + 12\beta_0 + \beta_1$

$g(10) = g(2(5) + 0) = 3g(5) + 25\varepsilon + 5\gamma + \beta_0$

$= 27\alpha + 46\varepsilon + 20\gamma + 10\beta_0 + 3\beta_1$

Finding General Form of CF

n	g(n)
1	$\alpha + 0\varepsilon + 0\gamma + 0\beta_0 + 0\beta_1$
2	$3\alpha + \varepsilon + \gamma + \beta_0 + 0\beta_1$
3	$3\alpha + \varepsilon + \gamma + 0\beta_0 + \beta_1$
4	$9\alpha + 7\varepsilon + 5\gamma + 4\beta_0 + 0\beta_1$
5	$9\alpha + 7\varepsilon + 5\gamma + 3\beta_0 + \beta_1$
6	$9\alpha + 12\varepsilon + 6\gamma + \beta_0 + 3\beta_1$
7	$9\alpha + 12\varepsilon + 6\gamma + 0\beta_0 + 4\beta_1$
8	$27\alpha + 37\varepsilon + 19\gamma + 13\beta_0 + 0\beta_1$

Finding General Form of CF

n	g(n)
9	$27\alpha + 37\varepsilon + 19\gamma + 12\beta_0 + \beta_1$
10	$27\alpha + 46\varepsilon + 20\gamma + 10\beta_0 + 3\beta_1$

Guessing the CF

$$g(n) = A(n)\alpha + B(n)\varepsilon + C(n)\gamma + D(n)\beta_0 + E(n)\beta_1$$

For all n ▯ N and $n \geq 1$, n can be written in the form $(2^k + h)$, where, 2^k is the highest power of 2, not exceeding n, and $(0 \leq h < 2^k)$. (Here $h \in N$). From our table we can observe that when $n = 2^k + h$, then the coefficient for $\alpha = 3^k$.

Example, let n=10, then $n = 2^3 + 2$, and the coefficient for α after computing g(n) is, $3^3 = 27$.

Exact CF Using Repertoire Method

CASE I: $g(n) = A(n)$

CASE II: $g(n) = 1$

CASE III: $g(n) = n$

x CASE IV: $g(n) = n^2$

x CASE V: $g(n) = n^3$

for all $n \in N$

CASE I: $g(n) = A(n)$

Let: $\varepsilon = \gamma = \beta_0 = \beta_1 = 0$ and $\alpha = 1$

In this case our general formula becomes:

$g(n) = A(n)1 + B(n)0 + C(n)0 + D(n)0 + E(n)0$

$g(n) = A(n)$

for all $n \in N$

CASE I: $g(n) = A(n)$

So, now the recurrence relation becomes:

$A(1) = \alpha$

$A(2n+ j) = 3A(n)$

for $j = 0,1$ and $n \geq 1$ and $n \in N$

[Recall: $g(2n+ j) = 3g(n) + \varepsilon n^2 + \gamma n + \beta_j$]

CASE I: g(n) = A(n)

Now we want to prove that:

$A(2^k + h) = 3^k\alpha$ for all $k \geq 0$

(Here $0 \leq h < 2k$ and $h \in N$)

It is easy to prove by induction that:

$A(2^k + h) = 3^k\alpha$ for all $k \geq 0$

CASE I: g(n) = A(n): Induction Proof

A(1) = 1, A(2n) = 3A(n), A(2n+1) = 3A(n), $A(n) = 3^k$ for $n = 2^k + h$ where $0 \le h < 2^k$

Base case: k=0 => n = 1 + h, h < 1 so n = 1

g(n) = A(n) and g(1) = 1 => A(1) = 1

$A(1) = 3^0 = 1$ [YES]

Assume $A(2^{k-1} + h) = 3^{k-1}$, $0 \le h < 2^{k-1}$

To prove: $A(2^k + h) = 3^k$, $0 \le h < 2^k$d

CASE I: g(n) = A(n): Induction Proof

Case 'n' even ($n = 2n_1$): $2^k + h = 2n_1$ iff h is even

$\Rightarrow n_1 = 2^{k-1} + h/2$

$A(2n_1) = A(2^k + h) = 3A(n_1) = 3A(2^{k-1} + h/2)$

$= 3 \cdot 3^{k-1} = 3^k$ [YES]

Case 'n' odd ($n = 2n_1 + 1$): $2^k + h = 2n_1 + 1$ iff h is odd

$\Rightarrow n_1 = 2^{k-1} + (h-1)/2$

$A(2n_1 + 1) = A(2^k + h) = 3A(n_1) = 3A(2^{k-1} + (h-1)/2)$

$= 3 \cdot 3^{k-1} = 3^k$ [YES]

CASE II: $g(n) = 1$

We have that $g(n)=1$, for all $n \in N$.
From $g(1) = \alpha$, we find that $g(1) = 1 = \alpha$. So $\alpha = 1$.

$$g(2n+0) = 3g(n) + \varepsilon n^2 + \gamma n + \beta_0$$
$$1 = 3(1) + \varepsilon n^2 + \gamma n + \beta_0$$
$$1 = 3 + \varepsilon n^2 + \gamma n + \beta_0$$
$$1 - 3 = \varepsilon n^2 + \gamma n + \beta_0$$
$$0 = \varepsilon n^2 + \gamma n + \beta_0 + 2, \text{ for all } n \in N$$

Hence, $\varepsilon = \gamma = 0$ and $\beta_0 = -2$

CASE II: $g(n) = 1$

$$g(2n+1) = 3g(n) + \varepsilon n^2 + \gamma n + \beta_1$$

$$1 = 3(1) + \varepsilon n^2 + \gamma n + \beta_1$$

$$1 = 3 + \varepsilon n^2 + \gamma n + \beta_1$$

$$1 - 3 = \varepsilon n^2 + \gamma n + \beta_1$$

$$0 = \varepsilon n^2 + \gamma n + \beta_1 + 2 \text{ , for all } n \in N$$

Hence, $\varepsilon = \gamma = 0$ and $\beta_1 = -2$

CASE II: $g(n) = 1$

Plugging the values of α, ε, γ, β_0 and β_1 into the general form of the CF:

$$g(n) = A(n)\alpha + B(n)\varepsilon + C(n)\gamma + D(n)\beta_0 + E(n)\beta_1$$

$$1 = A(n) - 2D(n) - 2E(n), \text{ for all } n \in N.$$

CASE III: $g(n) = n$

We have that $g(n) = n$, for all $n \in N$.
From $g(1) = \alpha$, we find that $g(1) = 1 = \alpha$. So $\alpha = 1$.

$$g(2n+0) = 3g(n) + \varepsilon n^2 + \gamma n + \beta_0$$
$$2n = 3n + \varepsilon n^2 + \gamma n + \beta_0$$
$$0 = 3n - 2n + \varepsilon n^2 + \gamma n + \beta_0$$
$$0 = \varepsilon n^2 + n + \gamma n + \beta_0$$
$$0 = \varepsilon n^2 + (\gamma+1)n + \beta_{0,}, \text{ for all } n \in N.$$

Hence $\gamma = -1$, $\varepsilon = \beta_0 = 0$

CASE III: $g(n) = n$

We have that $g(n)=n$, for all n

$g(2n+1) = 3g(n) + \varepsilon n^2 + \gamma n + \beta_1$

$2n + 1 = 3n + \varepsilon n^2 + \gamma n + \beta_1$

$1 = 3n - 2n + \varepsilon n^2 + \gamma n + \beta_1$

$1 = \varepsilon n^2 + n + \gamma n + \beta_1$

$0 = \varepsilon n^2 + (\gamma+1)n - 1 + \beta_1$, for all $n \in N$.

Hence, $\gamma = -1$, $\varepsilon = 0$, $\beta_1 = 1$

CASE III: $g(n) = n$

Plugging the values of α, ε, γ, β_o and β_1 into the general form of the CF:

$$g(n) = A(n)\alpha + +B(n)\varepsilon + C(n)\gamma + D(n)\beta_o + E(n)\beta_1$$

$$n = A(n) - C(n) + E(n) \text{ , for all } n \in N$$

CASE IV: $g(n) = n^2$

We have that $g(n) = n^2$, for all n.

From $g(1) = \alpha$, we deduce that $g(1) = 1^2 = 1 = \alpha$. So, $\alpha = 1$.

$$g(2n+1) = 3g(n) + \varepsilon n^2 + \gamma n + \beta_1$$
$$(2n+1)^2 = 3n^2 + \varepsilon n^2 + \gamma n + \beta_1$$
$$4n^2 + 1 + 4n = 3n^2 + \varepsilon n^2 + \gamma n + \beta_1$$
$$n^2 + 1 + 4n = \varepsilon n^2 + \gamma n + \beta_1$$
$$0 = (\varepsilon - 1)n^2 + (\gamma - 4)n + \beta_1 \text{, for all } n \in N.$$

Hence $\varepsilon = 1$, $\gamma = 4$, $\beta_1 = 1$

CASE IV: $g(n) = n^2$

$$g(2n+0) = 3g(n) + \varepsilon n^2 + \gamma n + \beta_0$$

$$(2n+0)^2 = 3n^2 + \varepsilon n^2 + \gamma n + \beta_0$$

$$4n^2 = 3n^2 + \varepsilon n^2 + \gamma n + \beta_0$$

$$0 = (\varepsilon - 1)n^2 + \gamma n + \beta_0$$

=> $\varepsilon = 1$, $\gamma = 0$, $\beta_0 = 0$

But for $g(2n+1)$ we just obtained:

$\varepsilon = 1$, $\gamma = 4$, $\beta_1 = 1$

Contradiction! Method fails!

CASE V: $g(n) = n^3$

We get a similar contradiction here too

(No need to proceed beyond first contradiction)

Binary Expansion Technique

Let us consider the case when the input value to R is of the form 2n+0 = 2n. Let the binary representation of 2n be:

$(b_m, b_{m-1}, \ldots., b_1, b_0)_2$

This means $2n = 2^m b_m + \ldots + 2b_1 + b_0$

We observe that $b_0 = 0$, because 2n is even

Dividing by 2:

$n = 2^{m-1} b_m + \ldots + b_1$ or $n = (b_m, b_{m-1}, \ldots., b_1)_2$

. . . Fact [1]

Binary Expansion Technique

Now, let us consider the case when the input value to R is of the form (2n+1). Let the binary representation of (2n+1) be:

$(b_m, b_{m-1}, \ldots., b_1, b_0)_2$

So, $(2n+1) = 2^m b_m + 2^{m-1} b_{m-1} + \ldots. + 2b_1 + b_0$

Since 2n+1 is an odd number $b_0 = 1$

Binary Expansion Technique

$2n+1 = 2^m b_m + \ldots + 2b_1 + b_0$

$2n+1 = 2^m b_m + \ldots + 2b_1 + 1$

$2n = 2^m b_m + \ldots + 2b_1$

$n = 2^{m-1} b_m + \ldots + b_1$

This means that $n = (b_m, \ldots, b_1)_2$ . . . Fact [2]

From [1] & [2], we observe same representation for both even/odd n. So, the CF need not have 2 cases

Binary Expansion Technique

$g(2n+0) = g(2n+1) = g(\ (b_m, b_{m-1}, \ldots., b_1, b_0)_2\ )$

$= 3g(\ (b_m, b_{m-1}, \ldots., b_1)_2\ ) + \varepsilon[(b_m, b_{m-1}, \ldots., b_1)_2]^2 + \gamma(b_m, b_{m-1}, \ldots., b_1)_2 + \beta_{bo}$

$= 3\{3g(\ (b_m, b_{m-1}, \ldots., b_2)_2\ ) + \varepsilon[(b_m, b_{m-1}, \ldots., b_2)_2]^2 + \gamma(b_m, b_{m-1}, \ldots., b_2)_2 + \beta_{b1}\} + \varepsilon[(b_m, b_{m-1}, \ldots., b_1)_2]^2 + \gamma(b_m, b_{m-1}, \ldots., b_1)_2 + \beta_{bo}$

OBSERVATION:

$\beta_{b_0} = \beta_1$ if $b_0 = 1$

And

$\beta_{b_0} = \beta_0$ if $b_0 = 0$

Similarly,

$\beta_{b_1} = \beta_1$ if $b_1 = 1$

And

$\beta_{b_1} = \beta_0$ if $b_1 = 0$

Binary Expansion Technique

$g\ (\ (b_m, b_{m-1}, \ldots., b_1, b_0)_2\ )$

$= 3^2 g(\ (b_m, b_{m-1}, \ldots., b_2)_2\ ) + 3\mathcal{E}[(b_m, b_{m-1}, \ldots., b_2)_2]^2 + 3\gamma(b_m, b_{m-1}, \ldots., b_2)_2 + 3\beta_{b1} + \mathcal{E}[(b_m, b_{m-1}, \ldots., b_1)_2]^2 + \gamma(b_m, b_{m-1}, \ldots., b_1)_2 + \beta_{bo}$

$= 3^2\{3g(\ (b_m, b_{m-1}, \ldots., b_3)_2\ ) + \mathcal{E}[(b_m, b_{m-1}, \ldots., b_3)_2]^2 + \gamma(b_m, b_{m-1}, \ldots., b_3)_2 + \beta_{b2}\} + 3\mathcal{E}[(b_m, b_{m-1}, \ldots., b_2)_2]^2 + 3\gamma(b_m, b_{m-1}, \ldots., b_2)_2 + 3\beta_{b1} + \mathcal{E}[(b_m, b_{m-1}, \ldots., b_1)_2]^2 + \gamma(b_m, b_{m-1}, \ldots., b_1)_2 + \beta_{bo}$

Binary Expansion Technique

(repeated)

$= 3^2\{3g((b_m, b_{m-1}, ..., b_3)_2) + \varepsilon[(b_m, b_{m-1}, ..., b_3)_2]^2 + \gamma(b_m, b_{m-1}, ..., b_3)_2 + \beta_{b2}\} + 3\varepsilon[(b_m, b_{m-1}, ..., b_2)_2]^2 + 3\gamma(b_m, b_{m-1}, ..., b_2)_2 + 3\beta_{b1} + \varepsilon[(b_m, b_{m-1}, ..., b_1)_2]^2 + \gamma(b_m, b_{m-1}, ..., b_1)_2 + \beta_{bo}$

$= 3^3g((b_m, b_{m-1}, ..., b_3)_2) + 3^2\varepsilon[(b_m, b_{m-1}, ..., b_3)_2]^2 + 3^2\gamma(b_m, b_{m-1}, ..., b_3)_2 + 3^2\beta_{b2} + 3\varepsilon[(b_m, b_{m-1}, ..., b_2)_2]^2 + 3\gamma(b_m, b_{m-1}, ..., b_2)_2 + 3\beta_{b1} + \varepsilon[(b_m, b_{m-1}, ..., b_1)_2]^2 + \gamma(b_m, b_{m-1}, ..., b_1)_2 + \beta_{bo}$

Binary Expansion Technique

(repeated)

$= 3^3 g((b_m, b_{m-1}, \ldots., b_3)_2) + 3^2 \varepsilon[(b_m, b_{m-1}, \ldots., b_3)_2]^2 + 3^2 \gamma(b_m, b_{m-1}, \ldots., b_3)_2 + 3^2 \beta_{b2} + 3\varepsilon[(b_m, b_{m-1}, \ldots., b_2)_2]^2 + 3\gamma(b_m, b_{m-1}, \ldots., b_2)_2 + 3\beta_{b1} + \varepsilon[(b_m, b_{m-1}, \ldots., b_1)_2]^2 + \gamma(b_m, b_{m-1}, \ldots., b_1)_2 + \beta_{bo}$

Generalizing:

$g((b_m, b_{m-1}, \ldots., b_1, b_o)_2)$

$= 3^m g((b_m)_2) + \varepsilon\{3^{m-1}[(b_m)_2]^2 + 3^{m-2}[(b_m b_{m-1})_2]^2 + 3^0[(b_m \ldots b_1)_2]^2\} + \gamma\{3^{m-1}(b_m)_2 + 3^{m-2}(b_m\ b_{m-1})_2 + \ldots + 3^0(b_m, \ldots., b_1)_2\} + 3^{m-1}\beta_{b_{m-1}} + 3^{m-2}\beta_{b_{m-2}} + \ldots. + 3^1\beta_{b1} + 3^0\beta_{bo}$

Binary Expansion Technique

Input value "1" is represented by only one bit "b_m". In this case $b_m = 1$.

Also, the only time when b_m is 0, is when the decimal number itself is 0. Any natural number is going to have $b_m = 1$.

Since the input to R are all natural numbers, therefore, in our computation $b_m = 1$.

So, $g(b_m) = g(1) = \alpha$

Binary Expansion Technique

$g\ ((b_m, b_{m-1}, \ldots, b_1, b_0)_2\ )$

$= 3^m\alpha + \varepsilon\{3^{m-1}[(b_m)_2]^2 + 3^{m-2}[(b_m b_{m-1})_2]^2 + 3^0[(b_m \ldots b_1)_2]^2\} + \gamma\{3^{m-1}(b_m)_2 + 3^{m-2}(b_m\ b_{m-1})_2 + \ldots + 3^0(b_m, \ldots, b_1)_2\} + 3^{m-1}\beta_{b_{m-1}} + 3^{m-2}\beta_{b_{m-2}} + \ldots + 3^1\beta_{b1} + 3^0\beta_{bo}$

$= 3^m\alpha + \varepsilon\{\Sigma 3^{m-i}[(b_m \ldots\ldots b_{m-i+1})_2]^2\} + \gamma\{\Sigma 3^{m-i}(b_m \ldots\ldots b_{m-i+1})_2\} + \Sigma 3^{m-i}\beta_{b_{m-i}}$

(i = 1 to m)

Example

$n = 8$: Binary representation: $(1000)_2$

$m = 3$, $b_0 = 0$, $b_1 = 0$, $b_2 = 0$ and $b_3 = b_m = 1$

CF: $g((b_m, b_{m-1}, \ldots, b_1, b_0)_2)$

$= 3^m\alpha + \varepsilon\{\Sigma 3^{m-i}[(b_m \ldots\ldots b_{m-i+1})_2]^2\} + \gamma\{\Sigma 3^{m-i}(b_m \ldots\ldots b_{m-i+1})_2\} + \Sigma 3^{m-i}\beta_{b_{m-i}}$

$g((1000)_2) = 3^3\alpha + \varepsilon\Sigma 3^{3-i}(b_3 \ldots\ldots b_{3-i+1})^2 + \gamma\Sigma 3^{3-i}(b_3 \ldots\ldots b_{3-i+1}) + \Sigma 3^{3-i}\beta_{b_{3-i}}$

Example

$g((1000)_2) = 3^3\alpha + \varepsilon\Sigma 3^{3-i}(b_3 \ldots\ldots b_{3-i+1})^2 + \gamma\Sigma 3^{3-i}(b_3 \ldots\ldots b_{3-i+1}) + \Sigma 3^{3-i}\beta_{b_{3-i}}$

$= 3^3\alpha + \varepsilon\{3^{3-1}[(b_3 \ldots b_{3-1+1})_2]^2 + 3^{3-2}[(b_3 \ldots b_{3-2+1})_2]^2 + 3^{3-3}[(b_3 \ldots b_{3-3+1})_2]^2\} + \gamma\{3^{3-1}(b_3 \ldots b_{3-1+1})_2 + 3^{3-2}(b_3 \ldots b_{3-2+1})_2 + 3^{3-3}(b_3 \ldots b_{3-3+1})_2\} + 3^{3-1}\beta_{b_{3-1}} + 3^{3-2}\beta_{b_{3-2}} + 3^{3-3}\beta_{b_{3-3}}$

$= 3^3\alpha + \varepsilon\{3^2[(1)_2]^2 + 3^1[(10)_2]^2 + 3^0[(100)_2]^2\} + \gamma\{3^2(1)_2 + 3^1(10)_2 + 3^0(100)_2\} + 3^2\beta_0 + 3^1\beta_0 + 3^0\beta_0$

Example

(repeated)

$= 3^3\alpha + \varepsilon\{3^2[(1)_2]^2 + 3^1[(10)_2]^2 + 3^0[(100)_2]^2\} + \gamma\{3^2(1)_2 + 3^1(10)_2 + 3^0(100)_2\} + 3^2\beta_0 + 3^1\beta_0 + 3^0\beta_0$

$= 27\alpha + \varepsilon\{9[1] + 3[4] + [16]\} + \gamma\{9(1) + 3(2) + 1(4)\} + 9\beta_0 + 3\beta_0 + \beta_0$

$= 27\alpha + 37\varepsilon + 19\gamma + 13\beta_0$

D	b_2	b_1	b_0
0	0	0	0
1	0	0	1
2	0	1	0
3	0	1	1
4	1	0	0
5	1	0	1
6	1	1	0
7	1	1	1

Finding General Form of CF

n	g(n)
1	$\alpha + 0\varepsilon + 0\gamma + 0\beta_0 + 0\beta_1$
2	$3\alpha + \varepsilon + \gamma + \beta_0 + 0\beta_1$
3	$3\alpha + \varepsilon + \gamma + 0\beta_0 + \beta_1$
4	$9\alpha + 7\varepsilon + 5\gamma + 4\beta_0 + 0\beta_1$
5	$9\alpha + 7\varepsilon + 5\gamma + 3\beta_0 + \beta_1$
6	$9\alpha + 12\varepsilon + 6\gamma + \beta_0 + 3\beta_1$
7	$9\alpha + 12\varepsilon + 6\gamma + 0\beta_0 + 4\beta_1$
8	$27\alpha + 37\varepsilon + 19\gamma + 13\beta_0 + 0\beta_1$