Closed Form for General Relaxed Radix Representation

Given Recursive Formula RF:

$$\begin{array}{ll} f(i) = \alpha_i, & i = 1, \cdots, d-1 \\ f(dn+j) = cf(n) + \beta_j, & n \geq 1,\ 0 \leq j < d \end{array}$$

Prove the following closed formula CF:

$$f((b_m, b_{m-1}, \cdots, b_1, b_0)_d) = (\alpha_{b_m}, \beta_{b_{m-1}}, \cdots, \beta_{b_1}, \beta_{b_0})_c$$

where β_{b_j} are defined by

$$\beta_{b_j} = \begin{cases} \beta_0 & b_j = 0 \\ \beta_1 & b_j = 1 \end{cases} ; \quad j = 0, ..., m-1,$$

Proof: First, we expand $(dn + j)$ on the basis of d, and derive the expansion for n.

$$dn + j = d^m b_m + d^{m-1} b_{m-1} + \cdots + d^1 b_1 + d^0 b_0, \quad (0 \leq j < d)$$

For this expansion, we must have $d^0 b_0 = j$.

$$dn + j = d^m b_m + d^{m-1} b_{m-1} + \cdots + d^1 b_1 + j, \quad (0 \leq j < d)$$

Then we have

$$dn = d^m b_m + d^{m-1} b_{m-1} + \cdots + d^1 b_1$$

$$then\ n = d^{m-1} b_m + d^{m-2} b_{m-1} + \cdots + d^0 b_1$$

We evaluate

$$\begin{array}{rcl} f(dn+j) & = & f((b_m, b_{m-1}, \cdots, b_0)_d) \\ & = & c \cdot f((b_m, b_{m-1}, \cdots, b_1)_d) + \beta_{b_0} \\ & = & c \cdot (c \cdot f((b_m, b_{m-1}, \cdots, b_2)_d) + \beta_{b_1}) + \beta_{b_0} \\ & = & c^2 \cdot f((b_m, b_{m-1}, \cdots, b_2)_d) + c \cdot \beta_{b_1} + \beta_{b_0} \\ & = & c^3 \cdot f((b_m, b_{m-1}, \cdots, b_3)_d) + c^2 \cdot \beta_{b_2} + c^1 \cdot \beta_{b_1} + c^0 \cdot \beta_{b_0} \\ & \vdots & \\ & = & c^m \cdot f((b_m)_d) + c^{m-1} \cdot \beta_{b_{m-1}} + c^{m-2} \cdot \beta_{b_{m-2}} + \cdots + c^1 \cdot \beta_{b_1} + c^0 \cdot \beta_{b_0} \\ & = & c^m \cdot \alpha_{b_m} + c^{m-1} \cdot \beta_{b_{m-1}} + c^{m-2} \cdot \beta_{b_{m-2}} + \cdots + c^1 \cdot \beta_{b_1} + c^0 \cdot \beta_{b_0} \\ & = & (\alpha_{b_m}, \beta_{b_{m-1}}, \cdots, \beta_{b_1}, \beta_{b_0})_c \end{array}$$

Hence we proved

$$f((b_m, b_{m-1}, \cdots, b_1, b_0)_d) = (\alpha_{b_m}, \beta_{b_{m-1}}, \cdots, \beta_{b_1}, \beta_{b_0})_c$$