FINITE and INFINITE SETS

Definition 1
A set \( A \) is FINITE iff there is a natural number \( n \in \mathbb{N} \) and there is a 1–1 function \( f \) that maps the set \( \{1,2,\ldots,n\} \) onto \( A \).

Definition 2
A set \( A \) is INFINITE iff it is NOT FINITE.

QUESTION 1
Use the above definitions to prove the following

FACT 1 A set \( A \) is INFINITE if and only if it contains a countably infinite subset, i.e. one can define a 1–1 sequence \( \{a_n\}_{n \in \mathbb{N}} \) of some elements of \( A \)

SOLUTION
S1. Proof of Implication

If \( A \) is infinite, then we can define a 1-1 sequence of elements of \( A \)

Let \( A \) be infinite. We define a sequence \( a_1, a_2, \ldots, a_n, \ldots \) as follows.

1. Observe that \( A \neq \emptyset \), because if \( A = \emptyset \), \( A \) would be finite. Contradiction. So there is an element of \( a \in A \).

   We define
   \[ a_1 = a \]

2. Consider a set \( A - \{a_n\} = A_1 \). \( A_1 \neq \emptyset \) because if \( A = \emptyset \), then \( A - \{a_1\} = \emptyset \) and \( A \) is finite. Contradiction.

   So there is an element \( a_2 \in A - \{a_1\} \) and \( a_1 \neq a_2 \).

   We defined
   \[ a_1, a_2 \]
   \[ a_1, a_2, \ldots, a_n, \ldots \] for \( a_1 \neq a_2 \neq \ldots \neq a_n \)

Assume that we defined a set \( A_n = A - \{a_1, \ldots, a_n\} \).

The set \( A_n \neq \emptyset \) because if \( A - \{a_1, \ldots, a_n\} = \emptyset \), then \( A \) is finite. Contradiction.

So there is an element
\[ a_{n+1} \in A - \{a_1, \ldots, a_n\} \]

and \( a_{n+1} \neq a_n \neq \ldots \neq a_1 \)

By mathematical induction, we have defined a 1-1 sequence
\[ a_1, a_2, \ldots, a_n, \ldots \]
elements of A.
2. Implication
If A contain a 1-1 sequence, then A is infinite.
Assume A is not infinite; i.e A is finite. Every subset of finite set is finite, so we can’t have a 1-1 infinite sequence of elements of A. Contradiction.

QUESTION 2 Use the above definitions and FACT 1 from QUESTION 1 the following characterization of infinite sets.

Dedekind Theorem A set A is INFINITE iff there is a set proper subset B of the set A such that |A| = |B|.

SOLUTION Part 1. If A is infinite, then there is B ⊊ A and

\[ f : A \xrightarrow{1-1} B \]

A is infinite, by Q1, we have a 1-1 sequence

\[ a_1, a_2, \ldots, a_n, \ldots \]

of elements A.
We take B = A − \{a_1\}, B ⊊ A and we define a function

\[ f : A \xrightarrow{1-1} B \]

as follows

\[ f(a_1) = a_2 \]
\[ f(a_2) = a_3 \]
\[ \vdots \]
\[ f(a_n) = a_{n+1} \]
\[ f(a) = a, \text{ for all other } a \in A \]

obviously, f is 1-1,onto
Observe: we have other choises of B!
Part 2. Assume that we have B ⊊ A are

\[ f : A \xrightarrow{1-1} B \]

We use Q1 to show that A is infinite; i.e we construct an 1-1 sequence \( a_1 \ldots a_n \) of elements of A as follows.
B ⊊ A, so A − B ≠ ∅ and we have \( b \in A − B \). This is our first element of the sequence.
Observe: \( f : A \xrightarrow{1-1} B \), so \( f(b) \in B \) and \( b \in A − B \), hence \( f(b) \neq b \) and \( f(b) \) is our second element of the sequence.
We have now, \( b, f(b) \neq b, b \in A − B, f(b) \in B \)
Take new, \( ff(b) \). As f is 1-1 and \( f(b) \neq b \), we get \( ff(b) \neq f(b) \neq b, ff(b) \in B \) and the sequence \( b, f(b), ff(b) \) is 1-1.
We create \( ff(b) = f^2(b) \)
We continue the construction by mathematical induction.
Assume that we have constructed a 1-1 sequence
\[
b, f(b), f^2(b), f^3(b), \ldots, f^n(b)
\]
Observe that \( ff^n(b) = f^{n+1}(b) \neq f^n(b) \) as \( f \) is 1-1.
By mathematical induction, we have that \( \{f^n(b)\}_{n \in \mathbb{N}} \) is a 1-1 sequence of elements of \( A \) and hence \( A \) is infinite.
QUESTION 3  Use technique from DEDEKIND THEOREM to prove the following

**Theorem**  For any infinite set $A$ and its finite subset $B$, $|A| = |A - B|$.

**SOLUTION**  A is infinite, then by Q1 there is a 1-1 sequence:

$$a_1, a_2, \ldots, a_n, \ldots$$

of elements of $A$.

Let $|B| = k$. We choose $k$ 1-1 sequences $\{c_n^j\}_{n \in N}$ of the sequence $\{a_n\}_{n \in N}$, such that $c_n^j \neq c_n^i$ for all $j \neq i, 1 \leq i, j \leq k$ and all $n \in N$.

Let $B = \{b_1, \ldots, b_k\}$. We construct a function $f : A \xrightarrow{1-1} A - \{b_1, \ldots, b_k\}$ as follows

$$f(b_1) = c_1^1, \quad f(c_1^1) = c_1^2, \ldots, f(c_n^1) = c_{n+1}^1$$

$$f(b_2) = c_1^2, \quad f(c_2^1) = c_2^2, \ldots, f(c_n^2) = c_{n+1}^2$$

$$\vdots$$

$$f(b_k) = c_1^k, \quad f(c_1^k) = c_2^k, \ldots, f(c_n^k) = c_{n+1}^k$$

$$f(a) = a \text{ all } a \in A - B$$

As all sequences $\{C_n^m\}_{n \in N, m=1,\ldots,k}$ are 1-1, and different, the function $f$ is 1-1 and obviously ONTO $A - B$.

QUESTION 4  Use DEDEKIND THEOREM to prove that the set $N$ of natural numbers is infinite.

**SOLUTION**  We use Dedekind theorem i.e we must define $f : N \xrightarrow{1-1} \mathbb{N} \not\subseteq N$. There are many such function for example $f(n) = n + 1, f : N \xrightarrow{1-1} N - \{0\}$

One can also use Q1 and define any 1-1 sequences in $N$.

QUESTION 5  Use DEDEKIND THEOREM to prove that the set $R$ of real numbers is infinite.

**SOLUTION**  We use Dedekind theorem

$$f(x) = 2^x \quad x \in R$$

$$f : R \xrightarrow{1-1} R^+$$

One can also use Q1 and define any 1-1 sequences in $R$.

QUESTION 6  Use technique from DEDEKIND THEOREM to prove that the interval $[a, b], a < b$ of real numbers is infinite and that $|[a, b]| = |(a, b)|$.

**SOLUTION**  Use construction in the proof of Q3.

$$f : [a, b] \xrightarrow{1-1} [a, b] - \{a, b\} = (a, b)$$

This is the solution I had in mine!
SOLUTION 2 Use Q3 \((a, b) = [a, b] - B, B : \text{finite}\)

QUESTION 7 Prove, using the above definitions 3 and 4 that for any cardinal numbers \(M, N, K\) the following formulas hold:

1. \(N \leq N\)
2. If \(N \leq M\) and \(M \leq K\), then \(N \leq K\).

SOLUTION

1. \(N \leq N\) means that for any set \(A\), \(|A| \leq |A|
\]
   \(f(a) = a\) establishes \(f : A \stackrel{1-1}{\longrightarrow} A\)
2. \(N \leq M\) and \(M \leq K\), then \(N \leq K\).
   We have \(|A| = N, |b| = M, |C| = K\) and \(f : A \stackrel{1-1}{\longrightarrow} B\) and \(g : B \stackrel{1-1}{\longrightarrow} C\), then we have to construct \(h : A \stackrel{1-1}{\longrightarrow} C\).
   \(h\) is a composition of \(f\) and \(g\). i.e \(h(a) = g(f(a))\), all \(a \in A\)

QUESTION 8 Prove, for any sets \(A, B, C\) the following holds.

Fact 2

If \(C \subseteq B \subseteq A\) and \(|A| = |C|\), then \(|A| = |B| = |C|\).

To prove \(|A| = |B|\) you must use definition 3, i.e to construct a proper function. Use the construction from proofs of Fact 1 and Question 3

SOLUTION

1. \(A, B, C\) are finite and \(|A| = |C|\), and \(C \subseteq B \subseteq A\), so \(A = B = C\), and have \(|A| = |B| = |C|\)
   2. \(A, B, C\) are infinite sets, we have \(|A| = |C|\) i.e we have \(f : A \stackrel{1-1}{\longrightarrow} C\)
   We want to construct a function \(g : A \stackrel{1-1}{\longrightarrow} B\), where \(A \subseteq B \subseteq C\)
   Take \(A - B\). We assume that \(A - B \neq \emptyset\), if not, \(A = B\), and \(|A| = |C|\) given \(|A| = |B| = |C|\).
   We consider case \(C \subset B \subset A\). Take any \(a \in (A - B)\), as \(f : A \stackrel{1-1}{\longrightarrow} C\), \(f(a) \in C\), \(f\) is 1-1 so \(ff(a) \neq f(a)\) . . . in general \(f^n(a) \neq f^{n+1}(a)\) and we have a sequence for any \(a \in A - B\)
   \(f(a), f^2(a), \ldots, f^n(a)\ldots\) of elements of \(C\).
   We construct a function \(g : A \stackrel{1-1}{\longrightarrow} B\)
   \[
   g(a) = f(a)
   \]
   \[
   g(f(a)) = f^2(a)
   \]
   \[
   g(f^2(a)) = f^3(a)
   \]
   \[
   \vdots
   \]
   \[
   g(f^n(a)) = f^{n+1}(a)
   \]
   \[
   g(x) = x \quad \text{for all other} \ x \in A
   \]

QUESTION 9 Prove the following
**Berstein Theorem** (1898) For any cardinal numbers $\mathcal{M}, \mathcal{N}$

$$\mathcal{N} \leq \mathcal{M} \text{ and } \mathcal{M} \leq \mathcal{N} \text{ then } \mathcal{N} = \mathcal{M}.$$ 

1. Prove first the case when the sets $A, B$ are disjoint.

2. Generalize the construction for 1. to the not-disjoint case.