cse547 Midterm 2 SOLUTIONS Fall 2023
(60pts + 5extra)

Problem 1 (10pts)

Evaluate $\sum_{k=1}^{n} k2^k$ by Method 5, i.e. by rewriting it as the multiple sum $\sum_{1\le j\le k\le n} 2^k$

Explain your steps. No full credit without explanation.

Solution

$$\sum_{k=1}^{n} k2^k$$
$$= \sum_{k=1}^{n} \left(\sum_{j=1}^{k} 1\right) 2^k$$
$$= \sum_{k=1}^{n} \sum_{j=1}^{k} 2^k$$
$$= \sum_{1\le j\le k\le n} 2^k$$
$$= \sum_{j=1}^{n} \sum_{k=j}^{n} 2^k$$
$$= \sum_{j=1}^{n} \left(\sum_{k=0}^{n} 2^k - \sum_{k=0}^{j-1} 2^k\right)$$
$$= \sum_{j=1}^{n} \left(2^{n+1} - 2^j\right)$$
$$= 2^{n+1} \sum_{j=1}^{n} 1 - \sum_{j=1}^{n} 2^j$$
$$= 2^{n+1}(n) - (2^{n+1} - 2)$$
$$= 2^{n+1}(n-1) + 2$$

Problem 2 (15pts)

Evaluate the sum $\sum_{k=1}^{n} \frac{2k+1}{k(k+1)}$ in two ways

1. (5 pts) Use Partial Fractions.

Hint : Represent $\frac{2k+1}{k(k+1)}$ by partial fractions; i.e. solve the equation
$\frac{2k+1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1}$
and represent the sum $\sum_{k=1}^{n} \frac{2k+1}{k(k+1)}$ as sum of corresponding two sums.

Solution

Evauate Partial Fractions

$$\frac{2k+1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1}$$

$$\frac{2k+1}{k(k+1)} = \frac{A(k+1)+Bk}{k(k+1)}$$

$$\frac{2k+1}{k(k+1)} = \frac{(A+B)k+A}{k(k+1)}$$

Comparing both the sides A = 1; A + B = 2; and A=1, B= 1

$$\frac{2k+1}{k(k+1)} = \frac{1}{k} + \frac{1}{k+1}$$

Evaluate the sum

$$\sum_{k=1}^{n}\left(\frac{1}{k} + \frac{1}{k+1}\right) = \sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{n} \frac{1}{k+1}$$

$$= H_n + H_{n-1} + \frac{1}{n+1}$$

$$= 2H_n - \frac{n}{n+1}$$

2. (10 pts) Use the Summation (integration) by parts: $\sum u \vartriangle v = uv - \sum Ev \vartriangle u$.

Hint $\frac{1}{k(k+1)} = (k-1)^{\underline{-2}}$ and $\sum x^{\underline{m}} \delta x = H_x$, iff $m = -1$

Solution By the Theorem 2 we get

$$\sum_{k=1}^{n} \frac{2k+1}{k(k+1)} = \sum_{k=1}^{n+1} \frac{2k+1}{k(k+1)} \delta k$$

$$\sum u \vartriangle v = uv - \sum Ev \vartriangle u$$

Let u(k) = 2k + 1;

$$\vartriangle u(k) = 2;$$

$$\vartriangle v(k) = \frac{1}{k(k+1)} = (k-1)^{\underline{-2}}$$

$$\text{v(k)} = -(k-1)^{\underline{-1}} = -\frac{1}{k}$$

$$Ev = \frac{-1}{k+1}$$

$\sum \frac{2k+1}{k(k+1)}\delta k = (2k+1)(\frac{-1}{k}) - \sum(\frac{-1}{k+1})2\ \delta k$

$= 2\sum k^{\underline{-1}}\ \delta k - \frac{2k+1}{k}$

$= 2H_k - 2 - \frac{1}{k} + C$

$[\sum x^{\underline{m}}\delta x = H_x,\ \text{if } m = -1]$

$\sum_{k=1}^{n+1} \frac{2k+1}{k(k+1)}\delta k = 2H_k - 2 - \frac{1}{k}\ |_1^{n+1}$

$= 2H_n + \frac{2}{n+1} - 2 - \frac{1}{n+1} - 2 + 2 + 1$

$= 2H_n + \frac{1}{n+1} - 1$

$= 2H_n - \frac{n}{n+1}$

Problem 3 (10pts)

1. (5 pts) Prove the formula $\Delta(c^{\underline{x}}) = \frac{c^{\underline{x+2}}}{(c-x)}$

Hint We define $\triangle(c^{\underline{x}}) = c^{\underline{x+1}} - c^{\underline{x}}$ for $c^{\underline{x+1}} = \underbrace{c(c-1)(c-2)\cdots(c-x+1)(c-x)}_{\text{x+1 factors}}$ and $c^{\underline{x}} = \underbrace{c(c-1)(c-2)\cdots(c-x+1)}_{\text{x factors}}$

Solution

We evaluate $\triangle(c^{\underline{x}}) = c^{\underline{x+1}} - c^{\underline{x}}$ for $c^{\underline{x+1}} = \underbrace{c(c-1)(c-2)\cdots(c-x)}_{\text{x+1 factors}}$ and $c^{\underline{x}} = \underbrace{c(c-1)(c-2)\cdots(c-x+1)}_{\text{x factors}}$

as follows

$$\begin{aligned}
\triangle(c^{\underline{x}}) &= (\mathbf{c}(\mathbf{c}-\mathbf{1})(\mathbf{c}-\mathbf{2})\cdots(\mathbf{c}-\mathbf{x}+\mathbf{1})(c-x)) - (\mathbf{c}(\mathbf{c}-\mathbf{1})(\mathbf{c}-\mathbf{2})\cdots(\mathbf{c}-\mathbf{x}+\mathbf{1})) \\
&= (\mathbf{c}(\mathbf{c}-\mathbf{1})(\mathbf{c}-\mathbf{2})\cdots(\mathbf{c}-\mathbf{x}+\mathbf{1})(c-x)) - (\mathbf{c}(\mathbf{c}-\mathbf{1})(\mathbf{c}-\mathbf{2})\cdots(\mathbf{c}-\mathbf{x}+\mathbf{1})) \\
&= (c(c-1)(c-2)\cdots(c-x+1))(c-x-1) \\
&= \frac{c(c-1)(c-2)\cdots(c-x+1)(\mathbf{c}-\mathbf{x})(c-x-1)}{(\mathbf{c}-\mathbf{x})} \\
&= \frac{c^{\underline{x+2}}}{(c-x)}
\end{aligned}$$

2. (5 pts) Prove that the property $-\triangle(f(x)) = \triangle(-f(x))$ holds for any function.

Solution

We evaluate

$$\begin{aligned}
-\triangle(f(x)) &= -(f(x+1) - f(x)) \\
&= -f(x+1) + f(X) \\
&= -f(x+1) - (-f(x)) \\
&= \triangle(-f(x))
\end{aligned}$$

Extra Credit Problem (5pts)

Use **1.** and **2.** from **Problem 3** and finite integration to prove that $\sum_{k=1}^{n} \frac{(-2)^{\underline{k}}}{k} = (-1) + (-1)^n n!$

Hint: Substitute c = -2 and x = x-2 in the formula **1.**$\Delta(c^{\underline{x}}) = \frac{c^{\underline{x+2}}}{(c-x)}$ and then apply property **2.** and finite integration.

Solution

We substitute c = -2 and x = x-2 in the formula **1.** $\Delta(c^{\underline{x}}) = \frac{c^{\underline{x+2}}}{(c-x)}$ and get

$$\begin{aligned}\triangle((-2)^{\underline{x-2}}) &= \frac{(-2)^{\underline{(x-2)+2}}}{(-2-(x-2))}\\ &= \frac{(-2)^{\underline{x}}}{-x}\\ &= -\frac{(-2)^{\underline{x}}}{x}\end{aligned}$$

By the above **2.** we get that

$$\begin{aligned}\triangle(-(-2)^{\underline{x-2}}) &= -\triangle((-2)^{\underline{x-2}})\\ &= --\frac{(-2)^{\underline{x}}}{x}\\ &= \frac{(-2)^{\underline{x}}}{x}\end{aligned}$$

By the Theorem 2

$\sum_{k=a}^{b-1} g(k) = \sum_{a}^{b} g(x)\delta x$ for all integers $b \geq a$.

and get

$$\sum_{k=1}^{n} \frac{(-2)^{\underline{k}}}{k} = \sum_{k=1}^{n+1} \frac{(-2)^{\underline{k}}}{k}\delta k \quad \text{for } n \geq 0.$$

We evaluate

$$\begin{aligned}\sum_{k=1}^{n+1} \frac{(-2)^{\underline{k}}}{k}\delta k &= -(-2)^{\underline{k-2}}\Big|_{1}^{n+1}\\ &= -(-2)^{\underline{n+1-2}} - [-(-2)^{\underline{1-2}}]\\ &= -(-2)^{\underline{n-1}} - [-(-2)^{\underline{-1}}]\\ &= -(-2)^{\underline{-1}} - (-2)^{\underline{n-1}}\\ &= \frac{1}{-2+1} - (-2)^{\underline{n-1}}\\ &= -1 - ((-2)(-2-1)(-2-2)\cdots(-2-(n-2)))\\ &= -1 - ((-2)(-3)(-4)\cdots(-n))\\ &= -1 + ((-1)(-2)(-3)\cdots(-n))\\ &= -1 + (-1)^n n!\end{aligned}$$

Problem 4 (10pts)

1. (5 pts) Prove that $\Sigma_{n=0}^{\infty} \frac{1}{(k+1)(k+2)} = 1$

Solution We evaluate

$$S_n = \Sigma_{k=0}^{n} \frac{1}{(k+1)(k+2)} = \Sigma_{k=0}^{n} k^{\underline{-2}} = \Sigma_{k=0}^{n+1} k^{\underline{-2}}\,\delta k = -\frac{1}{k+1}\Big|_0^{n+1} = -\frac{1}{n+2} + 1$$

We get, by definition

$$\Sigma_{n=0}^{\infty} \frac{1}{(k+1)(k+2)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} -\frac{1}{n+2} + 1 = 1$$

2. (5 pts) Use relevant Theorem and prove convergence of a proper infinite sum to prove that

$\lim_{n\to\infty} \frac{c^n}{n!} = 0$ for $c > 0$.

Solution

Evaluate:

$$\frac{a_{n+1}}{a_n} = \frac{c^{n+1}}{c^n} \frac{n!}{(n+1)!} = \frac{c}{n+1}$$

and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{c}{n+1} = 0 < 1 \text{ for } c > 0.$$

By D'Alambert Criterium $\sum_{n=1}^{\infty} \frac{c^n}{n!}$ converges.

By **Thm1**: If the infinite sum $\Sigma_{n=1}^{\infty} a_n$ **converges**, then $\lim_{n\to\infty} a_n = 0$

we get that $\lim_{n\to\infty} \frac{c^n}{n!} = 0$ for $c > 0$.

Problem 5 (15pts)

Here are 7 steps of our BOOK solution

1 $W = \sum_{n=1}^{1000} [n \text{ is a winner }] = \sum_{n=1}^{1000} \left[\lfloor \sqrt[3]{n} \rfloor \mid n\right]$

2 $W = \sum_{k,n} \left[k = \lfloor \sqrt[3]{n} \rfloor\right] [k|n] [1 \le n \le 1000]$

3 $W = \sum_{k,n,m} \left[k^3 \le n < (k+1)^3\right] [n = km] [1 \le n \le 1000]$

4 $W = 1 + \sum_{k,m} \left[k^3 \le km < (k+1)^3\right] [1 \le k < 10]$

5 $W = 1 + \sum_{k,m} \left[m \in \left[k^2 \ldots \frac{(k+1)^3}{k}\right)\right] [1 \le k < 10]$

6 $W = 1 + \sum_{1 \le k < 10} \left(\lceil k^2 + 3k + 3 + \frac{1}{k} \rceil - \lceil k^2 \rceil\right)$

7 $W = 1 + \sum_{1 \le k < 10} (3k + 4) = 1 + \frac{7 + 31}{2} 9 = 172$

Write detailed EXPLANATIONS of each of the following **transformations** steps

3. (4 pts), **4.** (7 pts), and **5.** (4 pts)

Solution LECTURE 11a