CSE541 Midterm 1 SOLUTIONS Spring 2011

QUESTION 1

Remark This question was designed to check if you have studied Exercise 0 and its solutions AND Midterm Practice Solutions!

1. Use Dedekind theorem to prove that the set R of real numbers is infinite.

Solution There are many $1-1$ functions that map R onto its proper subset, I repeat the same as I gave you in the solutions of Exercise 0: $f(x) = 2^x$.

2. Find a function f that is $1-1$ and maps R ONTO $R - \{1, 8, 10\}$.

Solution There are many ways to construct the function f, but all constructions must follow the one presented (general case in the solution of Exercise 0. Idea: you must define 3 DISJOINT, $1-1$ sequences a_n, b_n and c_n of real numbers and apply the definition of f from Exercise 0, Question 3.

For example, let $n \geq 1$

$$a_n = \frac{1}{n}, \quad b_n = 1 + \frac{1}{n}, \quad c_n = 2 + \frac{1}{n}.$$

We define f as follows:

$$f(1) = a_1, \quad f(a_n) = a_{n+1}, \quad \text{all} \quad n \geq 1,$$
$$f(8) = b_1 \quad f(b_n) = b_{n+1} \quad \text{all} \quad n \geq 1,$$
$$f(10) = c_1 \quad f(c_n) = c_{n+1} \quad \text{all} \quad n \geq 1,$$
$$\text{and} \quad f(x) = x \quad \text{for} \quad \text{all} \quad \text{other} \quad x \in R.$$

QUESTION 2 Here are some definitions; some of them are known to you and put as a reminder.

Definition 1 By a **m-valued semantics** S_m, $m \geq 2$ for a propositional language $\mathcal{L} = \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$ we understand any definition of of connectives $\neg, \cap, \cup, \Rightarrow$ as operations on a set $V_m = \{v_1, v_2, ...v_m\}$, $m \geq 2$ of logical values.
We assume that $v_1 \leq v_2 \leq ... \leq v_m$, i.e. V_m is totally ordered by a certain relation $\leq$ with v_1, v_m being smallest and greatest elements, respectively. We denote $v_1 = F$, $v_m = T$ and call them (total) False and Truth, respectively.

Example Let $m = 2$, $V_2 = \{F, T\}$, $F \leq T$. Semantics C_2 is called **a classical semantics** if it is defined by classical Truth Tables, i.e. we put, for any $a, b \in V_2$, $a \cup b = max\{a, b\}$, $a \cap b = min\{a, b\}$, $\neg T = F, \neg F = T$, and $a \Rightarrow b = \neg a \cup b$.

Definition 2 Let VAR be a set of propositional variables of $\mathcal{L}$ and let S_m be any m-valued semantics for $\mathcal{L}$. A truth assignment $v : VAR \longrightarrow V_m$ is called **a** S_m **model** for a formula A of $\mathcal{L}$ iff $v(A) = T$ and logical value $v(A)$ is evaluated accordingly to the semantics S_m. We denote is symbolically as

$$v \models_{S_m} A.$$

Any v such that v **is not a** S_m **model** for a formula A is called **a counter-model** for A.

Definition 3 A formula A of $\mathcal{L}$ is called **a** S_m **tautology** iff $v \models_{S_m} A$, for all v. We denote it by $\models_{S_m} A$, and $\models A$ for **classical semantics tautologies**.

Definition 4 A proof system S is **complete** with respect to a semantics S_m iff for any formula A,the following holds:

A is provable in S iff A is S_m tautology.

Q2 Part one

Remark This is a very straight forward problem; I wanted to stress the fact that connectives in extensional semantics are defined as ANY functions on proper sets of logical values and to remind you of the basic definitions of model, counter model, tautology in a given semantics even if very similar to classical s 2 valued semantics, are slightly diffract.

Let S_3 be a 3-valued semantics for $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$ defined as follows.
$V_3 = \{F, U, T\}$, for $F \leq U \leq T$ and

$\cup$	F	U	T
F	F	U	T
U	U	U	U
T	T	U	T

$\neg$	F	U	T
	T	F	U

$a \cap b = min\{a, b\}$, $\quad a \Rightarrow b = \neg a \cup b$, for any $a, b \in V_3$.

Consider the following classical tautologies:

$$A_1 = (A \cup \neg A), \qquad A_2 = (A \Rightarrow (B \Rightarrow A)).$$

(a) Find S_3 counter-models for A_1, A_2, if exist. Use shorthand notation.

Solution Any v such that $v^*(A) = v^*(B) = U$ is a counter-model for both, A_1 and A_2.

(b) Define a 2-valued semantics S_2 for $\mathcal{L}$, such that **none of** A_1, A_2 is a S_2 tautology. Verify your results. Use shorthand notation.

Solution This is not the only solution, but it is the simplest and most obvious! Here it is.

We define $\neg a = F, \quad a \Rightarrow b = F$ for all $a, b \in \{F, T\}$,and $a \cap b, a \cup b$ can be defined in the same, or ant other way, as these connectives do not appear in our formulas.

(c) Define a 3-valued semantics C_3 for $\mathcal{L}$, such that both A_1, and A_2 are a C_3 tautologies. Verify your results. Use shorthand notation.

Solution This is not the only solution, but it is the simplest and most obvious! Here it is (it follows the idea from (b).
We define $\neg a = F, \quad a \Rightarrow b = F$ for all $a, b \in \{F, T\}$, $a \cap b, a \cup b$ can be defined in the same, or any other way, as these connectives do not appear in our formulas.

We define $\neg a = T, \quad a \Rightarrow b = T$ for all $a, b \in \{F, U, T\}$, and $a \cap b, a \cup b$ can be defined in the same, or any other way, as these connectives do not appear in our formulas.

Q2 Part Two Let $S = (\mathcal{L}, \mathbf{A1}, \mathbf{A2}, \mathbf{A3}, MP)$ be a proof system with axioms:

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$,

The system S is **complete** with respect to classical semantics.

Verify whether S is complete with respect to 3-valued semantics S_3, as defined at the beginning of this question.

Solution (This is a kind of a "free points" problem!) System S is not complete because is not sound, as we have shown in (a).

QUESTION 3

Remark This problem is taken straight from the BOOK and your exercises solutions! I write the solution to spare your time!

Let S be from QUESTION 2, PART 2. The following Lemma holds in the system S.

LEMMA For any $A, B, C \in \mathcal{F}$,

(a) $(A \Rightarrow B), (B \Rightarrow C) \vdash_H (A \Rightarrow C)$,

(b) $(A \Rightarrow (B \Rightarrow C)) \vdash_H (B \Rightarrow (A \Rightarrow C))$.

Complete the proof sequence (in S)

$$B_1, ..., B_9$$

by providing comments how each step of the proof was obtained.

Solution

$B_1 =$ $(A \Rightarrow B)$

Hypothesis

$B_2 =$ $(\neg\neg A \Rightarrow A)$

Already PROVED

$B_3 =$ $(\neg\neg A \Rightarrow B)$

Lemma **a** for $A = \neg\neg A, B = A, C = B$

$B_4 =$ $(B \Rightarrow \neg\neg B)$

Already PROVED

$B_5 =$ $(\neg\neg A \Rightarrow \neg\neg B)$

Lemma **a** for $A = \neg\neg A, B = B, C = \neg\neg B$

$B_6 =$ $((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$

Already PROVED

B_7 $(\neg B \Rightarrow \neg A)$

B_5, B_6 and MP

B_8 $(A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$

$B_1 - B_7$

$B_9 =$ $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

Deduction Theorem

QUESTION 4

Remark This question is designed to check if you understand the notion of completeness, monotonicity, application of Deduction Theorem and use of some basic tautologies.

Consider any proof system S,

$$S = (\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}},\ AX,\ (MP)\frac{A, (A \Rightarrow B)}{B})$$

that is **complete** under classical classical semantics.

Definition 1 Let $X \subseteq F$ be any subset of the set F of formulas of the language $\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg\}}$ of S.

We define a set $Cn(X)$ of all **consequences** of the set X as follows

$$Cn(X) = \{A \in F :\ X \vdash_S A\},$$

i.e. $Cn(X)$ is the set of all formulas that can be proved in S from the set $(AX \cup X)$.

Q4 Part 1

(i) Prove that for any subsets X, Y of the set F of formulas the following **monotonicity property** holds.

If $X \subseteq Y$, then $Cn(X) \subseteq Cn(Y)$

Solution Let $C \in Cn(X)$, it means that $X \vdash_S C$, i.e C has a formal proof from $X \cup AX$, but $X \subseteq Y$, hence also this proof is also a proof from $Y \cup AX$, i.e $Y \vdash_S C$ and $C \in Cn(Y)$. This proves that $Cn(X) \subseteq Cn(Y)$.

(ii) Prove that for any $X \subseteq F$, the set $\mathbf{T}$ of all propositional classical tautologies is a subset of $Cn(X)$, i.e.

$$\mathbf{T} \subseteq Cn(X).$$

Solution The proof system S is complete, i.e $\mathbf{T} = \{A :\ \vdash_S A\}$. By definition of the consequence, $\{A \in F :\ \vdash_S A\} = Cn(\emptyset)$ and hence by completeness, $Cn(\emptyset) = \mathbf{T}$.

For any set X, $\emptyset \subseteq X$ so by monotonicity proved in **(i)**,

$$\mathbf{T} \subseteq Cn(X).$$

Part two Prove that for any $A, B \in F,\ X \subseteq F$,

$$(A \cap B) \in Cn(X) \text{ iff } A \in Cn(X) \text{ and } B \in Cn(X)$$

Hint: Use the Monotonicity and Completeness of S i.e. the fact that any tautology you might need is provable in S.

Solution Assume $(A \cap B) \in Cn(X)$, i.e. $X \vdash_S (A \cap B)$.

From Monotonicity (Part 1), completeness, and the fact that $\models ((A \cap B) \Rightarrow A), \models ((A \cap B) \Rightarrow B)$ we get that

$$X \vdash_S ((A \cap B) \Rightarrow A), \quad X \vdash_S ((A \cap B) \Rightarrow B).$$

We have hence the following.

$$X \vdash_S (A \cap B), \quad \text{(assumption)},$$
$$X \vdash_S ((A \cap B) \Rightarrow A) \quad \text{(completeness, monotonicity)},$$
$$X \vdash_S A, \quad \text{MP}.$$

Similarly, $X \vdash_S (A \cap B)$, (assumption), $X \vdash_S ((A \cap B) \Rightarrow B)$ (completeness, monotonicity), and so we get $X \vdash_S B$ by MP.

Hence $A \in Cn(X)$ and $B \in Cn(X)$.

Assume now that $A \in Cn(X)$ and $B \in Cn(X)$. I.e. $X \vdash_S A$, and $X \vdash_S B$.

From Completeness of S and the fact that $\models (A \Rightarrow (B \Rightarrow (A \cap B)))$, and monotonicity (proved in Part 1) we get that

$$X \vdash_S (A \Rightarrow (B \Rightarrow (A \cap B))).$$

From above we have the following:

$$X \vdash_S A \quad \text{(assumption)}, \quad X \vdash_S B \quad \text{(assumption)},$$
$$X \vdash_S (A \Rightarrow (B \Rightarrow (A \cap B))) \quad \text{(completeness, monotonicity)},$$
$$X \vdash_S (B \Rightarrow (A \cap B)) \ \text{MP}, \quad X \vdash_S (A \cap B) \ \text{MP}.$$

This proves that $(A \cap B) \in Cn(X)$.

Part Three: Prove that for any $A, B \in F$,

$$Cn(\{A, B\}) = Cn(\{(A \cap B)\})$$

Hint: Use Deduction Theorem and Completeness of S.

Solution Assume $C \in Cn(\{A, B\})$, i.e. $\{A, B\} \vdash_S C$, what we usually write as $A, B \vdash_S C$.

By Deduction Theorem applied twice we get that

$$\vdash_S (A \Rightarrow (B \Rightarrow C)).$$

We use completeness of S, monotonicity and the fact that

$$\models (((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C)))$$

to construct the following.

$$\vdash_S (A \Rightarrow (B \Rightarrow C)) \quad \text{(assumption and Deduction Theorem)},$$
$$\vdash_S (((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))), \text{(completeness, monotonicity)},$$
$$\vdash_S ((A \cap B) \Rightarrow C) \quad \text{MP},$$
$$(A \cap B) \vdash_S C, \quad \text{(Deduction Theorem)}.$$

i.e. we have proved that $C \in Cn(\{(A \cap B)\}) = Cn(A \cap B)$.

Assume now that $C \in Cn(\{(A \cap B)\})$, i.e. $(A \cap B) \vdash_S C$.

By Deduction Theorem, $\vdash_S ((A \cap B) \Rightarrow C)$.

We want to prove that $C \in Cn(\{A, B\})$, what is equivalent, by the Deduction Theorem applied twice to proving that

$$\vdash_S (A \Rightarrow (B \Rightarrow C)).$$

The proof as in the previous case. We use completeness of S, and the fact that

$$\models (((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)))$$

to get that $\vdash_S (A \Rightarrow (B \Rightarrow C))$ as follows.

$$\vdash_S ((A \cap B) \Rightarrow C) \quad \text{(assumption and Deduction Theorem)},$$
$$\vdash_S (((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)), \quad \text{(completeness)},$$

$$\vdash_S (A \Rightarrow (B \Rightarrow C)), \quad \text{MP},$$

what ends the proof.

QUESTION 5 Given a tautology A, and the set V_A of all truth assignment restricted to A, the Proof 1 of the Completeness Theorem for the system S defines a method of efficiently combining $v \in V_A$ to construct a proof of A in S.

Let consider the following tautology $A = A(a, b)$

$$A = ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a))$$

Write down all steps of the construction of the proof of A as described in the Proof 1 with justification why they are correct.

Solution The proof follows EXACTLY the Proof 1 and the Solution of Question 6 from PRACTICE MIDTERM for the shorter CASE of $A = A(a, b)$. **This question was designed just to check if you did study the Solutions of the Practice Midterm!** not to mention the Proof 1, as I have asked you to do. I repeat the solution here - as yet another exercise.

By the Main Lemma and the assumption that $\models A(a, b)$ any $v \in V_A$ defines formulas B_a , B_b such that

$$B_a, B_b \vdash A. \tag{1}$$

The proof is based on a method of using all $v \in V_A$ (there is 4 of them) to define a process of elimination of all hypothesis B_a, B_b in (1) to construct the proof of A in S i.e. to prove that $\vdash A$.

Step 1: elimination of B_b.

Observe that by definition, B_b is b or $\neg b$ depending on the choice of $v \in V_A$. We choose two truth assignments $v_1 \neq v_2 \in V_A$ such that

$$v_1|\{a\} = v_2|\{a\} \tag{2}$$

and $v_1(b) = T$ and $v_2(b) = F$.

Case 1: $v_1(b) = T$, by definition $B_b = b$. By the property (2), assumption that $\models A$, and the Main Lemma applied to v_1

$$B_a, b \vdash A.$$

By Deduction Theorem we have that

$$B_a \vdash (b \Rightarrow A). \tag{3}$$

Case 2: $v_2(b) = F$ hence by definition $B_b = \neg b$. By the property (2), assumption that $\models A$, and the Main Lemma applied to v_2

$$B_a, \neg b \vdash A.$$

By the Deduction Theorem we have that

$$B_a \vdash (\neg b \Rightarrow A). \tag{4}$$

By the assumed provability of the formula A8 for $A = b, B = A$ we have that

$$\vdash ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A)).$$

By monotonicity we have that

$$B_a \vdash ((b \Rightarrow A) \Rightarrow ((\neg b \Rightarrow A) \Rightarrow A)). \tag{5}$$

Applying Modus Ponens twice to the above property (5) and properties (3), (4) we get that

$$B_a \ \vdash \ A. \tag{6}$$

and hence we have eliminated B_b.

Step 2: elimination of B_a from (6). We repeat the Step 1.

As before we have 2 cases to consider: $B_a = a$ or $B_a = \neg a$. We choose from V_A two truth assignments $g_1 \neq g_2 \in V_A$ such that

$$g_1(a) = T \text{ and } g_2(a) = F. \tag{7}$$

Case 1: $g_1(a) = T$, by definition $B_a = a$. By the property (7), assumption that $\models A$, and the Main Lemma applied to g_1

$$a \ \vdash \ A.$$

By Deduction Theorem we have that

$$\vdash \ (a \Rightarrow A). \tag{8}$$

Case 2: $g_2(a) = F$ hence by definition $B_a = \neg a$. By the property (7), assumption that $\models A$, and the Main Lemma applied to g_2

$$\neg a \ \vdash \ A.$$

By the Deduction Theorem we have that

$$\vdash \ (\neg a \Rightarrow A). \tag{9}$$

By the assumed provability of the formula (**??**) for $A = a, B = A$ we have that

$$\vdash ((a \Rightarrow A) \Rightarrow ((\neg a \Rightarrow A) \Rightarrow A)). \tag{10}$$

Applying Modus Ponens twice to the above property (10) and properties (8), (9) we get that

$$\vdash \ A. \tag{11}$$

and hence we have eliminated B_a and B_b.